An introduction to Brakke flows

Masterclass “Recent Progress on Singularity Analysis and Applications of the Mean Curvature Flow”

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Contents

1 Preface

These are notes on background material for one course of the Masterclass "Recent Progress on Singularity Analysis and Applications of the Mean Curvature Flow" at the Copenhagen Centre for Geometry & Topology in April/May 2024.

We omit many fundamental and important results on smooth mean curvature flow but focus on an introduction to one of the weak notions of mean curvature flow, known as Brakke flows. We have freely copied from the lectures notes

- B. White, *Topics in mean curvature flow*, lecture notes by O. Chodosh. Available at http://web.stanford.edu/~ochodosh/notes.html

of the beautiful course by Brian White, with some further simplifications to adjust to the format. So we claim in no way originality.

Here is a list of further introductory texts on mean curvature flow:

- K. Ecker, *Regularity Theory for Mean Curvature Flow*, Birkhäuser
- C. Mantegazza, *Lecture Notes in Mean Curvature Flow*, Progress in Mathematics, Volume 290, Birkhäuser
- R. Haslhofer, *Lectures on mean curvature flow.* Available at https://arxiv.org/abs/1406.7765.
- R. Haslhofer, *Lectures on mean curvature flow of surfaces.* Available at https://arxiv.org/abs/2105.10485

2 Geometry of Hypersurfaces

We give an introduction to the geometry of hypersurfaces in Euclidean space. For a more detailed background, we recommend [12, Chapter 6] and [24, §7].

We restrict ourselves to manifolds of codimension 1 in an Euclidean ambient space, i.e. we consider a n-dimensional smooth manifold M, without boundary, either closed or complete and non-compact and an immersion (or embedding)

$$F : M \to \mathbb{R}^{n+1}.$$

We call the image $F(M)$ a hypersurface. We will often identify points on M with their image under the immersion, if there is no risk of confusion.

Let $x = (x_1, \dots, x_n)$ be a local coordinate system on M. The components of a vector v in the given coordinate system are denoted by v^i, the ones of a covector w are w_i. Mixed tensors have components with upper and lower indices depending on their type. We denote by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle_e$$

the induced metric on M, where $\langle \cdot, \cdot \rangle_e$ is the Euclidean scalar product on $\mathbb{R}^{n+1}$. Note that the metric g induces anatural isomorphism between the tangent and the cotangent space. In coordinates, this is expressed in terms of raising/lowering indexes by means of the matrcies g_{ij} and g^{ij}, where g^{ij} is the inverse of g_{ij}. The scalar product on the tangent bundle naturally extends to any tensor bundle. For instance the scalar product of two $(1,2)$-tensors T^i_{jk} and S^i_{jk} is defined by

$$\langle T^i_{jk}, S^i_{jk} \rangle = T_i^{jk} S^i_{jk} = T^l_{pq} S^i_{jk} g_{li} g^{pj} g^{qk} \,.$$

The norm of a tensor T is then given by $|T| = \sqrt{\langle T, T \rangle}$. The volume element $d\mu$ (which is just the restriction of the n-dimensional Hausdorff measure to M), is given in local coordinates by

$$d\mu = \sqrt{\det g_{ij}}\, dx$$

Recall that on the ambient space $\mathbb{R}^{n+1}$ we have the standard covariant derivative $\bar{\nabla}$ given via directional derivatives of each coordinate, i.e. for two smooth vectorfields on

X, Y on $\mathbb{R}^{n+1}$ we have

$$\bar{\nabla}_X Y\Big|_p = (D_{X(p)}Y^1(p), \cdots, D_{X(p)}Y^{n+1}(p))$$

where $Y(p) = (Y^1(p), \cdots, Y^{n+1}(p))$, and $D_{X(p)}$ is the directional derivative at p in direction $X(p)$. Recall that to define $D_{X(p)}Y^i(p)$ it is only necessary to locally know Y along an integral curve to X through p. Given two vectorfields V, W along $F(M)$ and tangent to M we thus define the connection

$$\nabla_V W := (\bar{\nabla}_V W)^T ,$$

where ^{T} is the projection to the tangent space of M. One can check that this is the Levi-Civita connection corresponding to the induced metric g. In coordinates we obtain for the derivative of a vector v^i or a covector w^i the formulas

$$\nabla_k v^i = \frac{\partial v^i}{\partial x_k} + \Gamma^i_{jk} v^j, \qquad \nabla_k w_j = \frac{\partial w_j}{\partial x_k} - \Gamma^i_{jk} w_i,$$

where Γ^i_{jk} are the Christoffel symbols of the the connection ∇. This covariant derivative extends to tensors of all kind, in coordinates, we have e.g. for a (1,2)-tensor T^i_{jl}:

$$\nabla_k T^i_{jl} = \frac{\partial T^i_{jl}}{\partial x_k} + \Gamma^i_{mk} T^m_{jl} - \Gamma^m_{jk} T^i_{ml} - \Gamma^m_{kl} T^i_{jm} \ ,.$$

If f is a function, we set $\nabla_k f = \frac{\partial f}{\partial x_k}$, which concides with the differential $df\Big(\frac{\partial}{\partial x_k}\Big)$. Using the isomorphism induced by the metric g we can regard ∇f also as element of the tangent space, in this case it is called the *gradient* of f. The gradient of f can be identified with a vector in $\mathbb{R}^{n+1}$ via the differential dF; such a vector is called the *tangential gradient* of f and is denoted by $\nabla^M f$, given in coordinates by

$$\nabla^M f = \nabla^i f \frac{\partial F}{\partial x_i} = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial F}{\partial x_i} \ .$$

The word "tangential" comes from the equivalent definition of $\nabla^M f$ in case f is a function defined on the ambient space $\mathbb{R}^{n+1}$. It can be checked that $\nabla^M f$ is the projection of the standard Euclidean gradient DF onto the tangent space of M, that is

$$\nabla^M f = Df - \langle Df, \nu\rangle_e \, \nu$$

where ν is a local choice of unit normal to M.

For two tangential vectorfields V, W, the shape operator is given by

$$S_V W = (\bar{\nabla}_V W)^\perp$$

where $^\perp$ is the projection to the normal space of M. Thus we have

$$\bar{\nabla}_V W = \nabla_V W + S_V W \ .$$

For local choice of unit normal vector field ν, the second fundamental form of M, a $(0,2)$-tensor, is given by

$$A(V,W) = -\langle S_V W, \nu \rangle_e = \langle W, \bar{\nabla}_V \nu \rangle_e \ ,$$

or in coordinates $A = (h_{ij})$ by

$$h_{ij} = -\left\langle \frac{\partial^2 F}{\partial x_j \partial x_i}, \nu \right\rangle_e = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j} \nu \right\rangle_e \ .$$

The matrix of the Weingarten map $W(X) = \bar{\nabla}_X \nu : T_pM \to T_pM$ is given by $h^i_{\ j} = g^{il} h_{lj}$. The *principal curvatures* of M at a point are the eigenvalues of the symmetric matrix h^i_j, or equivalently the eigenvalues of h_{ij} with respect to g_{ij}. We denote the principal curvatures by $\lambda_1 \leq \cdots \leq \lambda_n$. The *mean curvature* is defined as the trace of the second fundamental form, i.e.

$$H = h^i_{\ i} = g^{ij} h_{ij} = \lambda_1 + \ldots + \lambda_n \ .$$

The square of the norm of the second fundamental form will be denoted by

$$|A|^2 = g^{mn} g^{st} h_{ms} h_{nt} = h^n_{\ s} h^s_{\ n} = \lambda_1^2 + \ldots + \lambda_n^2.$$

It is easy to see that $|A|^2 \geq H^2/n$, with equality only if all the curvatures coincide; in fact we have the identity

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j} (\lambda_i - \lambda_j)^2 \ . \tag{2.1}$$

Clearly, A, W, H depend on the choice of orientation; if ν is reversed, their sign changes. But note that the *mean curvature vector*

$$\mathbf{H} = -H\nu$$

is independent of the orientation; in particular it is well defined globally even if M is non-orientable.

We will call a hypersurface *convex* if the principal curvatures are non-negative everywhere. Observe that, with these definitions, if $F(M)$ is the boundary of a convex set, and the normal is outward pointing, then all principal curvatures are non-negative.

Recall the curvature tensor

$$R(X,Y,Z,W) = g(\nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W, Z)$$

for vectorfields X, Y, Z, W on M.The Gauss equations relate the Riemann w.r.t. g to the curvature tensor of the ambient space in terms of the second fundamental form. Since the Euclidean ambient space is flat, we obtain

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}\,.$$

Thus the scalar curvature is given by

$$R = g^{ik}g^{jl}R_{ijkl} = H^2 - |A|^2 = 2\sum_{i<j} \lambda_i \lambda_j\ .$$

We also recall the *Codazzi equations*, which say that

$$\nabla_i h_{jk} = \nabla_j h_{ik}\,, \qquad i,j,k \in \{1,\dots,n\},$$

i.e. taking into account the symmetry of h_{ij}, this implies that the tensor $\nabla A = \nabla_i h_{jk}$ is totally symmetric.

Let $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, i.e. an ambient vectorfield with compact support. Let $(\phi_t)_{-\varepsilon<t<\varepsilon}$ be the associated family of diffeomorphisms, i.e.

$$\frac{\partial \phi_t}{\partial t} = X(\phi_t)\,, \qquad \phi_0 = \mathrm{id}\ .$$

We then obtain a one-parameter family of variations of $F(M)$ via $\phi_t(F(M)$. We compute the variation of the measure as

$$\begin{aligned} \frac{\partial d\mu}{\partial t}\Big|_{t=0} &= \frac{\partial \sqrt{\det g_{ij}}}{\partial t}\Big|_{t=0} dx = \frac{1}{\sqrt{\det g_{ij}}} (\det g_{ij}) g^{rs} \left\langle \frac{\partial X}{\partial x_r}, \frac{\partial F}{\partial x_s} \right\rangle_e dx \\ &= g^{rs} \left\langle \bar\nabla_{\frac{\partial F}{\partial x_r}} X, \frac{\partial F}{\partial x_s} \right\rangle_e d\mu\,, \end{aligned} \tag{2.2}$$

which leads us to define the tangential divergence

$$\mathrm{div}_M X = g^{ij} \left\langle \bar\nabla_{\frac{\partial F}{\partial x_i}} X, \frac{\partial F}{\partial x_j} \right\rangle_e = \sum_{i=1}^n \langle \bar\nabla_{e_i} X, e_i \rangle_e$$

where $e_1, \cdots, e_n$ is an ON-basis of T_pM. Recall the divergence theorem on a closed manifold

$$\int_M \operatorname{div}_M(X)\, d\mu = 0\,, \tag{2.3}$$

for $X \in \mathrm{Vec}_c(M)$. This follows directly from Stokes' theorem. For the normal part of a non-tangential vector field, one obtains

$$\begin{aligned}\operatorname{div}_M(X^\perp) &= \operatorname{div}_M(\langle X,\nu\rangle_e\,\nu) = \langle \nabla^M\langle X,\nu\rangle_e,\nu\rangle_e + \langle X,\nu\rangle_e \operatorname{div}_M\nu\\ &= \langle X,\nu\rangle_e g^{ij}\left\langle \bar\nabla_{\frac{\partial F}{\partial x_i}}\nu, \frac{\partial F}{\partial x_j}\right\rangle_e = \langle X,\nu\rangle_e g^{ij}h_{ij} = \langle X,\nu\rangle_e H = -\langle X,\mathbf{H}\rangle_e\end{aligned}$$

Together with (2.3) this yields the general divergence theorem

$$\int_M \operatorname{div}_M(X)\, d\mu = \int_M \operatorname{div}_M(X^T) + \operatorname{div}_M(X^\perp)\, d\mu = -\int_M \langle X,\mathbf{H}\rangle_e\, d\mu\,, \tag{2.4}$$

for $X \in \mathrm{Vec}_c(\mathbb{R}^{n+1})$. Together with (2.2) this yields the first variation formula

$$\frac{\partial}{\partial t}\bigg|_{t=0}\int_{\phi_t(M)} 1\, d\mu_t = \int_M \operatorname{div}_M(X)\, d\mu = -\int_M \langle X,\mathbf{H}\rangle_e\, d\mu\,. \tag{2.5}$$

We recall the *Laplace-Beltrami operator* on functions $f: M \to \mathbb{R}$ given by

$$\Delta^M f = \operatorname{div}_M(\nabla^M f)\ .$$

We write simply Δ instead of Δ^M. One can easily check that

$$\Delta^M f = g^{ij}\nabla_i\nabla_j f = g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k}\right) = \frac{1}{\sqrt{\det g_{ij}}}\frac{\partial}{\partial x_i}\left(\sqrt{\det g_{ij}}\, g^{ij}\frac{\partial f}{\partial x_j}\right)\ .$$

The divergence theorem then gives the usual integration by parts formula

$$\int_M f\Delta h\, d\mu = -\int_M \langle \nabla f, \nabla h\rangle\, d\mu = \int_M h\Delta f\, d\mu\ .$$

If f is a function on the ambient space we have by the above calculations

$$\begin{aligned}\Delta^M f &= \operatorname{div}_M(\nabla^M f) = \operatorname{div}_M(Df) - \operatorname{div}_M(Df^\perp)\\ &= \Delta^{\mathbb{R}^{n+1}} f - D^2 f(\nu,\nu) + \langle Df, \mathbf{H}\rangle_e\ .\end{aligned} \tag{2.6}$$

Thus Δ^M not only neglects the contribution of the second derivatives normal to M, but also takes into account the curvature of M.

Let $X = (x_1,\dots,x_{n+1})$ be the coordinates of $\mathbb{R}^{n+1}$. Equation (2.6) yields

$$\Delta^M x_i = \langle \mathbf{H}, e_i\rangle_e$$

where e_i is the i-th basis vector of $\mathbb{R}^{n+1}$. We can thus write

$$\Delta^M X = \mathbf{H}\ .$$

Note that in coordinates the vectorfield X is just given by F, and we can write

$$\Delta^M F = \mathbf{H}\ .$$

We also note the identity

$$\Delta^M |X|_e^2 = 2n + 2\langle X, \mathbf{H}\rangle_e\,. \tag{2.7}$$

The second fundamental form corresponds in a certain sense to second derivatives of an immersion, and its symmetry reflects that second partial derivatives of a function commute. Similarly the Codazzi equations can be seen as a geometric manifestation that third partial derivatives commute. Thus we can also expect that there is a symmetry of the second covariant derivatives of the second fundamental form. This identity is known as *Simon's identity*:

$$\nabla_k\nabla_l h_{ij} = \nabla_i\nabla_j h_{kl} + h_{kl}h_i^{\ m}h_{mj} - h_{km}h_{il}h^m_{\ j} + h_{kj}h_i^{\ m}h_{ml} - h_k^{\ m}h_{ij}h_{ml} \tag{2.8}$$

For a proof see [19]. We note the following two consequences

$$\Delta h_{ij} = \nabla_i\nabla_j H + Hh_i^{\ m}h_{mj} - h_{ij}|A|^2 \tag{2.9}$$

and

$$\frac{1}{2}\Delta|A|^2 = h^{ij}\nabla_i\nabla_j H + |\nabla A|^2 + H\mathrm{tr}(A^3) - |A|^4\ . \tag{2.10}$$

We give the explicit expressions of the main geometric quantities in the case when $F(M)$ is the graph of a function $x_{n+1} = u(x_1, \ldots, x_n)$. We choose the orientation where ν points downwards. By straightforward computations one gets

$$\nu = \frac{(D_1u, \ldots, D_nu, -1)}{\sqrt{1+|Du|^2}}\ , \tag{2.11}$$

$$g_{ij} = \delta_{ij} + D_iuD_ju, \qquad g^{ij} = \delta_{ij} - \frac{D_iuD_ju}{1+|Du|^2}\,, \tag{2.12}$$

$$h_{ij} = \frac{D^2_{ij}u}{\sqrt{1+|Du|^2}}, \qquad H = \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\ , \tag{2.13}$$

where div is the standard divergence on $\mathbb{R}^n$.

We also recall the strong maximum principle scalar functions:

Theorem 2.1 (Strong maximum principle for parabolic equations)**.** *Let* (M, g_t) *be a closed Riemannian manifold with a smooth family of metrics* $(g_t)_{t\in[0,T)}$ *and* $f : M \times [0,T) \to \mathbb{R}$ *satisfying*

$$\frac{\partial f}{\partial t} \geq \Delta f + b^i \nabla_i f + c\, f$$

for some smooth funtions b^i, c, *where* $c \geq 0$. *If* $f(\cdot, 0) \geq 0$ *then*

$$\min_M f(\cdot, t) \geq \min_M f(\cdot, 0)\,.$$

Furthermore, if $f(p, t_0) = \min_M f(\cdot, 0)$ *for some* $p \in M,\ t > 0$, *then* $f \equiv \min_M f(\cdot, 0)$ *for* $0 \leq t \leq t_0$.

For a proof see for example [13, Chapter 6.4 and Chapter 7.1.4] .

3 Basic properties

Let M^n be closed (or non-compact and complete), and $F : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth family of immersions. Let $M_t := F(M,t)$. We call this family a mean curvature flow starting at an initial immersion F_0, if

$$\begin{aligned} \frac{\partial F}{\partial t} &= -H \cdot \nu = \mathbf{H} \quad (= \Delta_{M_t} F) \\ F(\cdot, 0) &= F_0 \ . \end{aligned} \tag{3.1}$$

Remark 3.1: i) In general, it suffices to ask that

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \mathbf{H}\,.$$

One solves the ODE on M given by

$$\frac{\partial \phi}{\partial t} = -dF^{-1}\left(\left(\frac{\partial F}{\partial t}\right)^{T}\right)(\phi)$$

with $\phi(0) = \mathrm{id}$. Then $\tilde{F} := F \circ \phi$ solves usual MCF.

ii) The evolution equation for a surface, which is locally given as the graph of a function u, is thus

$$\left(\frac{\partial u}{\partial t}\, e_{n+1}\right)^{\perp} = \mathbf{H}$$

or equivalently

$$\frac{\partial u}{\partial t}\, \langle e_{n+1}, \nu \rangle = -H \ ,$$

which yields

$$\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2}\ \mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1 + |Du|^2}\right) D_i D_j u\,. \tag{3.2}$$

This is a quasilinear parabolic equation.

iii) By formula (2.4) we have for an evolution with normal speed $-f\nu$ that

$$\frac{d}{dt}|M_t| = \frac{d}{dt}\int_M 1\,d\mu_t = -\int_M fH\,d\mu\,,$$

and thus for mean curvature flow

$$\frac{d}{dt}|M_t| = -\int_M |\mathbf{H}|^2\,d\mu_t\,.$$

By the Hölder's inequality, mean curvature flow decreases area the fastest, when comparing with speeds with the same L^2-norm. Furthermore, along mean curvature flow one has the natural estimate

$$\int_0^T\int |\mathbf{H}|^2\,d\mu_t\,dt = |M_0| - |M_T| \leq |M_0|\,.$$

Examples: There are not many explicit examples of mean curvature flow solutions.

i) The most basic one is the evolution of a sphere with initial radius $R > 0$. Assuming that the solutions remains rotationally symmetric (which follows from uniqueness, see later), we obtain the following ODE for the radius $r(t)$:

$$\frac{\partial r}{\partial t} = -\frac{n}{r}.$$

with initial condition $r(0) = R$. Integrating yields $r(t) = \sqrt{R^2 - 2nt}$. Note that the maximal existence time $T = R^2/(2n)$ is finite and the curvature blows up for $t \to T$. Furthermore, the shrinking sphere is an example of a solution which only moves by scaling, a so-called *self-similar shrinker*.

By the previous example the evolution of a cylinder

$$\mathbb{S}^k_R \times \mathbb{R}^{n-k}$$

remains cylindrical with radius given by $r(t) = \sqrt{R^2 - 2kt}$. Note that again this solution is *self-similarly shrinking*.

Another class of examples are translating solutions. Assuming that they translate with speed one in direction τ, they satisfy the elliptic equation

$$H = -\langle \tau, \nu\rangle.$$

Assuming that the solution is graphical, i.e. $x_{n+1} = u(x_1, \cdots, x_n)$, and moving in e_{n+1} direction we obtain from (3.2) that it satisfies the equation

$$\left(\delta^{ij} - \frac{D^i u\, D^j u}{1 + |Du|^2}\right) D_i D_j u = 1\,.$$

In one dimension the equation becomes

$$y_{xx} = 1 + y_x^2$$

which can be integrated explicitly, yielding $y(x) = -\ln\cos x$ for $|x| < \pi/2$, up to translation and adding constants. This solution is usually called the *grim reaper*.
In higher dimensions it can be shown that there is a unique, convex, rotationally symmetric solution - but which is defined on the whole space. For properties of this solution see [8]. For $n = 2$ these are the unique convex translating entire graphs, but for $n \geq 3$ there exist entire convex translating graphs which are not rotationally symmetric, see [25].

The upwards translating grim reaper given by $e^{-y(t)} = e^{-t}\cos x(t)$ and the downwards translating grim reaper given by $e^{y(t)} = e^{-t}\cos x(t)$ can be combined to give another pair of solutions given implicitly as the solution set of

$$\cosh y(t) = e^t \cos x(t)\ , \tag{3.3}$$

and

$$\sinh y(t) = e^t \cos x(t)\ . \tag{3.4}$$

The *paperclip*, given as solution of (3.3) restricted to $|x| < \pi/2$ desribes a compact *ancient* solution that for $t \to 0$ becomes extinct in a round point and for $t \to -\infty$ looks like two copies of the grim reaper glued together smoothly. The *hairclip* (3.4) is an *eternal* solution, which for $t \to -\infty$ looks like an infinite row of grim reapers, alternating between translating up and translating down, and for $t \to +\infty$ converges to a horizontal line.

We have the following short-time existence result.

Theorem 3.2 (Short-time existence)**.** *Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed n-dimensional manifold M. Then there exists a unique smooth solution on a* *maximal time interval* $[0, T)$ *for* $T \in (0, \infty]$.

The difficulty to prove this result comes from the geometric nature of the flow, which makes any solution invariant under diffeomorphisms of M and thus the evolution equation is only weakly parabolic. There different ways to prove this result. One can either follow the approach of Hamilton [17] for the Ricci flow and use the Nash-Moser Implicit function theorem. Alternatively one can use the so-called De Turck trick to break the diffeomeorphism invariance. The maybe most natural way for mean curvature flow is to write the evolving surfaces $M_t = F(M, t)$ for a short time as an exponential normal graph over $M_0 = F_0(M)$. One can then check that the height function u satisfies a quasilinear parabolic equation similar to (3.2) for which standard results for those type of equations can be applied. For details see [19].

The strong maximum principle implies the following.

Theorem 3.3 (Avoidance principle)**.** *Assume two solutions to mean curvature flow* $(M_t^1)_{t\in[0,T)}$ *and* $(M_t^2)_{t\in[0,T)}$ *are initially disjoint (and at least one of them is compact), i.e.* $M_0^1 \cap M_0^2 = \emptyset$. *Then* $M_t^1 \cap M_t^2 = \emptyset \quad \forall\, t \in (0, T)$.

Proof. Assume that this is not the case. Then there exists a first time $t_0 \in (0, T)$ where $M_{t_0}^1$ and $M_{t_0}^2$ touch at the point $x_0 \in \mathbb{R}^{n+1}$. Note that this implies that $T_{x_0}M_{t_1}^1 = T_{x_0}M_{t_1}^2 := T$ and there is an $\varepsilon > 0$ such that we can write $(M_t^1)_{t_0-\varepsilon\le t\le t_0}$ and $(M_t^2)_{t_0-\varepsilon\le t\le t_0}$ locally as graphs over the affine space $x_0 + T$. The two graph functions u_1, u_2 satisfy (3.2) which we write as

$$\frac{\partial u}{\partial t} = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1 + |Du|^2}\right) D_{ij}u =: a^{ij}(Du) D_{ij}u.$$

We can assume w.l.o.g that $u_2 \le u_1$ and $u_1 = u_2$ at (x_0, t_0). But note that $v = u_1 - u_2$

satisfies a linear parabolic equation:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= a^{ij}(Du_1)D_iD_ju_1 - a^{ij}(Du_2)D_iD_ju_2 \\
&= \int_0^1 \frac{d}{ds}\big(a^{ij}(D(su_1+(1-s)u_2)D_{ij}(su_1+(1-s)u_2)\big)\,ds \\
&= \bigg(\int_0^1 a^{ij}(D(su_1+(1-s)u_2))\,ds\bigg)D_{ij}v \\
&\qquad + \bigg(\int_0^1 \frac{\partial a^{ij}}{\partial p_k}(D(su_1+(1-s)u_2))D_{ij}(su_1+(1-s)u_2)\,ds\bigg)D^kv \\
&=: \tilde{a}^{ij}D_{ij}v + \tilde{b}^kD_kv\,,
\end{aligned}$$

where p is the Du variable of $a^{ij}(p)$. Note that $\tilde{a}^{ij}$ is symmetric and strictly positve. Since $v \geq 0$ and $v = 0$ at (x_0, t_0) the strong maximum principle implies that $v \equiv 0$ which yields a contradiction. □

With more or less the same argument one can show the following.

Corollary 3.4 (Preservation of embeddedness)**.** *If M_0 is closed and embedded, then M_t is embedded for all t.*

Remark 3.5: (i) Enclosing a compact initial hypersurface M_0 by a large sphere, and using that the maximal existence time of the evolution of the sphere is finite, we obtain that the maximal existence time T is finite.

(ii) Note the we can translate a solution to mean curvature flow in the ambient space and get a new solution to mean curvature flow. Thus the avoidance principle implies that the distance between two disjoint solutions is non-decreasing in time.

(iii) In case M_0 is embedded, we will always choose ν to be the *outward* unit normal.

4 Weak compactness for submanifolds

To understand compactness for mean curvature flow, we see that the evolution equation for the measure gives a natural bound on the space-time integral of H^2. We are thus naturally led to trying to take limits of submanifolds under some weak curvature bounds. To be concrete, suppose that M_i is a sequence of m-submanifolds in $\mathbb{R}^N$. Assume that the areas of M_i are locally uniformly bounded. We may later assume that

$$\int_{M_i} |\mathbf{H}|^2 \, d\mu_i$$

is also locally uniformly bounded, but this will not be important in the beginning. We ask if it is possible to understand a weak limit of the M_i.

The simplest possibility is as follows: note that any m-submanifold M determines a Radon measure μ_M by

$$\mu_M(S) = \mathcal{H}^m(M \cap S)\,.$$

Equivalently, for any compactly supported continuous function f, we set

$$\int f \, d\mu_M := \int_M f \, d\mathcal{H}^m\,.$$

Hence, for the M_i as before, we may pass to a subsequence so that $\mu_{M_i} \rightharpoonup \mu$ weakly.

This is quite a coarse procedure (as we will see later), and we would like a more refined definition. An important observation is that M actually defines a Radon measure on $G(m,N)$, where $G(m,N)$ is the Grassmanian of m-dimensional subspaces in $\mathbb{R}^N$. We define the measure V_M:

$$\int f \, dV_M = \int_M f(x, Tan(M,x)) \, d\mathcal{H}^m$$

for $f : \mathbb{R}^N \times G(m,N) \to \mathbb{R}$ a continuous function of compact support, where $Tan(M,x)$

is the tangent space of M at x. Alternatively, we have

$$V_M(S) = \mathcal{H}^m(\{x \in M : (x, Tan(M,x)) \in S\}) .$$

We can take a subsequence so that $V_{M_i} \rightharpoonup V$. Note that for $\pi : \mathbb{R}^N \times G(m,N) \to U$ the natural projection map, we can check that $\pi_* V_M = \mu_M$.

Definition 4.1. *An m-dimensional* varifold *in $\mathbb{R}^N$, is a Radon measure on $\mathbb{R}^N \times G(m,N)$.*

4.1 The pushforward of a varifold

We remark here that the pushforward by a C^1 compactly supported diffemorphism $f : U \to U$ does not respect the measures μ_M: i.e., it may be that $f_\# \mu_M \neq \mu_{f(M)}$. For example, if M is a circle, and if f shrinks the circle to a smaller radius, $f_\# \mu_M$ will have the same total mass as μ_M, but $\mu_{f(M)}$ will not. However, we can easily define the pushforward of a varifold $f_\# V$ and check that $f_\# V_M = V_{f(M)}$ (this is just the change of variable formula with the Jacobian).

4.2 Integral varifolds

The class of general varifolds will be way to general and will include numerous pathological examples. We thus would like to define a smaller class.

Lemma 4.2. *Suppose that M, M' are m-dimensional C^1-submanifolds of $\mathbb{R}^N$. Let*

$$Z = \{x \in M \cap M' : Tan(M,x) \neq Tan(M',x)\}.$$

Then, $\mathcal{H}^m(Z) = 0$.

Proof. If M, M' are hypersurfaces, then Z is an $(m-1)$-dimensional C^1-submanifold by transversality. In higher co-dimension, one may show that $M \cap M'$ is contained in a $(m-1)$-dimensional C^1-submanifold, by projecting to a lower dimensional space. □

Corollary 4.3. *Suppose that* $S \subset \cup_i M_i$ *and* $S \subset \cup_i M_i'$ *for* M_i, M_i', *m-dimensional* C^1*-submanifolds of* $\mathbb{R}^N$. *Define* T *on* S *by*

$$T(x) = Tan(M_i, x)\,,$$

where i *is the first* i *so that* $x \in M_i$, *i.e.,* $x \in M_i \setminus \cup_{j<i} M_j$. *Define* T' *similarly. Then*

$$T(x) = T'(x)$$

for a.e. $x \in S$.

Thus, for a S a Borel subset of a C^1-submanifold $M \subset \mathbb{R}^N$, we define V_S to be the varifold given by

$$\int f\, dV_S = \int_S f(x, \mathrm{Tan}(M, x))\, d\mathcal{H}^m\,.$$

This does not depend on the choice of M, by the previous corollary. This allows us to give the following two equivalent definitions.

Definition 4.4. *An* integral m-varifold *is a varifold which can be written as*

$$V = \sum_{i=1}^{\infty} V_{S_i}\,.$$

Definition 4.5. *Suppose that* $\theta \in \mathscr{L}^1_{loc}(U; \mathbb{Z}^+; \mathcal{H}^m)$ *and* $S = \{\theta > 0\} \subset Z \cup (\cup_i M_i)$ *where* Z *has* $\mathcal{H}^m(Z) = 0$ *and the* M_i *are* m*-dimensional* C^1*-submanifolds of* U. *This data defines an* integral m-varifold V_θ *by*

$$\int f\, dV_\theta = \int_S f(x, T(x))\theta(x)\, d\mathcal{H}^m(x) = \sum_i \int_{M_i \setminus \cup_{j<i} M_j} f(x, \mathrm{Tan}(M_i, x))\theta(x)\, d\mathcal{H}^m\,.$$

Note to relate the two definitions, we can easily see that $\theta = \sum \mathbf{1}_{S_i}$.

4.3 First variation of a varifold

First we recall

Theorem 4.6 (Divergence theorem)**.** *Suppose that* M *is a* C^2*-submanifold of* $\mathbb{R}^N$ *and*

X a compactly suppported C^1-vectorfield. We set

$$\mathrm{div}_M X = \sum_i \langle \nabla_{e_i} X, e_i \rangle \,,$$

for $e_1, \cdots, e_m$ an orthonormal basis for $\mathrm{Tan}(M,x)$. Then

$$\begin{aligned}\int_M \mathrm{div}_M X &= \int_M \mathrm{div}_M X^\perp + \int_M \mathrm{div}_M X^T \\ &= -\int_M \langle X, \mathbf{H} \rangle \, d\mathcal{H}^m + \int_{\partial M} \langle X, \mathbf{n} \rangle \, d\mathcal{H}^{m-1} \,,\end{aligned}$$

where $\mathbf{n}$ is the exterior unit conormal to ∂M.

Note that the left hand side makes sense if M is just C^1. If there is a distributional vector field H making this true, then we say that H is the *weak mean curvature.*

Now, for V an m-varifold, we define the *first variation* of V by

$$\delta V(X) = \int \mathrm{div}_T X \, dV(x,T) \,,$$

where

$$\mathrm{div}_T X = \sum_i \langle \nabla_{e_i} X, e_i \rangle$$

for $e_1, \cdots, e_m$ an orthonormal basis for T. If V is an integral varifold, this can be written as

$$\delta V(X) = \int \mathrm{div}_{T(V,x)} X \, d\mu_V \,.$$

Remark 4.7: If X is a C^1-vectorfield with support in $K \subset U$, K compact, and $(\phi_t)_{-\varepsilon<t<\varepsilon}$ the associated 1-parameter family of diffeomorphims with $\phi_0 = \mathbf{id}$, then one can show (see for example L. Simon's lecture notes, [24]), that

$$\frac{d}{dt}\Big|_{t=0} \big(((\phi_t)_\# V)(K) \big) = \delta V(X)$$

as in the smooth case. (This just follows from expanding the Jacobian).

Note that trivially, if $V_i \rightharpoonup V$, then $\delta V_i(X) \to \delta V(X)$. Suppose now that we have local bounds on the first variation in the form

$$|\delta V(X)| \le C_K \|X\|_0$$

for supp$X \subset K \Subset \mathbb{R}^N$. Then the Riesz represantation theorem implies that there is a Radon measure λ on $\mathbb{R}^N$ and a λ-measurable unit vectorfield Λ such that

$$\delta V(X) = \int \langle X, \Lambda \rangle \, d\lambda \,.$$

Decomposing λ with respect to μ_V, there is $\lambda_{\mathrm{ac}} \ll \mu_V$ and λ_{sing} so that

$$\begin{aligned} \delta V(X) &= \int \langle X, \Lambda \rangle \, d\lambda_{\mathrm{ac}} + \int \langle X, \Lambda \rangle \, d\lambda_{\mathrm{sing}} \\ &= \int \left\langle X, \underbrace{\Lambda \frac{d\lambda_{\mathrm{ac}}}{d\mu_V}}_{=:-\mathbf{H}} \right\rangle d\mu_V + \int \langle X, \underbrace{\Lambda}_{=:\mathbf{n}} \rangle \, d\lambda_{\mathrm{sing}} \end{aligned}$$

Thus, in the case that V has locally bounded first variation, we have

$$\delta V(X) = - \int \langle X, \mathbf{H} \rangle \, d\mu_V + \int \langle X, \mathbf{n} \rangle \, d\lambda_{\mathrm{sing}} \,.$$

The following deep theorem due to Allard [1] is the reason reason that the class of integral varifolds is a reasonable one to study.

Theorem 4.8 (Allard's compactness theorem)**.** *Suppose that $V_i \rightharpoonup V$ is a sequence of integral varifolds converging weakly to a varifold V. If the V_i have locally uniformly bounded first variation, i.e., for $K \Subset \mathbb{R}^N$ there is C_K independent of i such that for all C^1-vectorfields X with supp $X \subset K$, we have*

$$|\delta V_i(X)| \le C_K \|X\|_0$$

then V is also an integral varifold.

Note that in the theorem, we trivially obtain the bounds

$$|\delta V(X)| \le C_K \|X\|_0$$

For example, a sequence of hypersurfaces satisfy the hypothesis of Allard's theorem if and only if

$$\int_K |\mathbf{H}_i| \, d\mu_{M_i} + \int_K d\sigma_i \leq C_K \,,$$

where $d\sigma_i$ is the boundary measure for M_i.

Example 4.9: Note that the quantities "$|\mathbf{H}|$" and "$d\sigma$" can be "mixed up" in the limit. For example consider a sequence of ellipses converging to a line with multiplicity two. Note that $\int |\mathbf{H}_i| = 2\pi$ and $\sigma_i = 0$, but in the limit, $\mathbf{H} = 0$ but $\sigma \neq 0$. Conversely, a sequence of polygons converging to a circle has $\mathbf{H} = 0$, but nontrivial boundary measure (at the vertices), but the circle only has mean curvature and no boundary.

Theorem 4.10. *Suppose that for V_i integral varifolds, we have $V_i \rightharpoonup V$ and that V_i has locally bounded first variation and no generalized bounday. Equivalently, we are assuming that*

$$\delta V_i(X) = -\int \langle X, \mathbf{H}_i \rangle \, d\mu_i \,.$$

Assume that

$$\int_K |\mathbf{H}_i|^2 \, d\mu_i \leq C_K < \infty$$

for $K \Subset U$. Then

(1) V is an integral varifold.

(2) We have

$$\int \langle \mathbf{H}_i, X \rangle \, d\mu_i \to \int \int \langle \mathbf{H}, X \rangle \, d\mu \,,$$

where X is a continuous vectorfield with compact support in U.

We note that any L^p for $p > 1$ could replace L^2 here.

Proof. We first note that $V_i \rightharpoonup V$ implies local mass bounds. Thus, local bounds for $\mathbf{H}_i$ in L^2 imply local bounds in L^1, and V is an integral varifold, by Allard's theorem. Note that

$$\delta V_i(X) = -\int_K \langle \mathbf{H}_i, X \rangle \leq \left(\int_K |\mathbf{H}_i|^2 \right)^{\frac{1}{2}} \left(\int_K |X|^2 \right)^{\frac{1}{2}} \leq C_K^{\frac{1}{2}} \left(\int_K |X|^2 \right)^{\frac{1}{2}} \,.$$

Thus,

$$|\delta V(X)| \leq C_K^{\frac{1}{2}} \left(\int_K |X|^2 \right)^{\frac{1}{2}} .$$

From this, it follows from the Riesz representation theorem that $\mathbf{H} \in \mathscr{L}^2_{\text{loc}}(d\mu_V)$. Since we thus have local uniform bounds in L^2 for $\mathbf{H}_i$ and $\mathbf{H}$, for the statement in (2) we can approximate any continuous vectorfield with compact support on U by a C^1-vectorfield with compact support on U. The stated convergence for a C^1-vectorfield then follows by the definition of of the first variation and the convergence $V_i \rightharpoonup V$. $\square$

5 Brakke flow

We now discuss Brakke's weak mean curvature flow [3, 22], we follow here the conventions used in [29].

Definition 5.1. *An (n-dimensional)* integral Brakke flow *in* $\mathbb{R}^{n+1}$ *is a* 1*-parameter family of Radon measures* $(\mu(t))_{t\in I}$ *over an interval* $I \subset \mathbb{R}$ *so that:*

1. *For almost every* $t \in I$*, there exists an integral* n*-dimensional varifold* $V(t)$ *with* $\mu(t) = \mu_{V(t)}$ *so that* $V(t)$ *has locally bounded first variation and has mean curvature* $\mathbf{H}$ *orthogonal to* $Tan(V(t), \cdot)$ *almost everywhere.*

2. *For a bounded interval* $[t_1, t_2] \subset I$ *and any compact set* $K \subset \mathbb{R}^{n+1}$*,*

$$\int_{t_1}^{t_2} \int_K (1 + |\mathbf{H}|^2) d\mu(t) dt < \infty.$$

3. *If* $[t_1, t_2] \subset I$ *and* $f \in C_c^1(\mathbb{R}^{n+1} \times [t_1, t_2])$ *has* $f \geq 0$ *then*

$$\int f(\cdot, t_2)\, d\mu(t_2) - \int f(\cdot, t_1)\, d\mu(t_1) \leq \int_{t_1}^{t_2} \int \left(-|\mathbf{H}|^2 f + \langle \mathbf{H}, \nabla f\rangle + \tfrac{\partial}{\partial t} f\right)\, d\mu(t)\, dt.$$

We will often write $\mathcal{M}$ *for a Brakke flow* $(\mu(t))_{t\in I}$*, with the understanding that we're referring to the family* $I \ni t \mapsto \mu(t)$ *of measures satisfying Brakke's inequality.*

Remark 5.2: We note that if M_t is a smooth mean curvature flow, then

$$\begin{aligned}\frac{d}{dt}\int_{M_t} f\, dA &= \int_{M_t} \left(-\langle \mathbf{H}, v\rangle f + \langle \nabla^\perp f, v\rangle + \frac{\partial f}{\partial t}\right) dA \\ &= \int_{M_t} \left(-|\mathbf{H}|^2 f + \langle \nabla f, \mathbf{H}\rangle + \frac{\partial f}{\partial t}\right) dA\end{aligned}$$

where the first equality holds for any smooth flow with velocity v. An obvious question is why we require the inequality, rather than the equality in the definition of Brakke flow. The reason for this is that only the inequality is possibly preserved under limits. For example, the weak limit of rescaled grim reapers is a multiplicity two line for $t < 0$ and is empty for $t > 0$!

Theorem 5.3. *Suppose that (μ_t) is an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$. Let $\phi = (r^2 - |x|^2 - 2nt)^+$. Then*

$$\int \phi^4 \, d\mu_t$$

is decreasing in t.

Proof. For $f = \frac{1}{4}\phi^4$, we compute

$$\nabla f = \phi^3 \nabla \phi \,,$$

so

$$\mathrm{div}_M(\nabla f) = 3\phi^2 |\nabla^T \phi|^2 + \phi^3 \mathrm{div}_M(\nabla \phi) \geq -2n\phi^3 \,.$$

Moreover,

$$\frac{\partial f}{\partial t} = \phi^3 \frac{\partial \phi}{\partial t} = -2n\phi^3 \,.$$

Thus, using f as a test function in the definition of Brakke flow yields

$$\begin{aligned} \int f \, d\mu_b - \int f \, d\mu_a &\leq \int_a^b \int \left(-|\mathbf{H}|^2 + \langle \mathbf{H}, \nabla f \rangle + \frac{\partial f}{\partial t} \right) d\mu_t \, dt \\ &\leq \int_a^b \int \left(-\mathrm{div}_M(\nabla f) + \frac{\partial f}{\partial t} \right) d\mu_t \, dt \\ &\leq 0 \,, \end{aligned}$$

as desired. □

Corollary 5.4. *For an integral n-dimensional Brakke flow μ_t on $\mathbb{R}^{n+1}$ defined on $[a, b]$, for K compact, we have uniform mass bounds, i.e., there is c_K independent of t, so that*

$$\mu_t(K) \leq c_K < \infty$$

for $t \in [a, b]$.

Theorem 5.5. *An integral Brakke flow satisfies*

$$\int_a^b \int_K |\mathbf{H}|^2 \, d\mu_t \, dt < C_K(1+b-a)$$

for any $K \Subset \mathbb{R}^{n+1}$.

Proof. Suppose that $\phi \in C_c^2, \phi \geq 0$, is time independent. Recall that $\frac{|\nabla \phi|^2}{\phi} \leq C(|\nabla^2 \phi|)$ (Exercise). Hence, since

$$\langle \nabla \phi, \mathbf{H} \rangle \leq \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} + \frac{1}{2} \phi |\mathbf{H}|^2 \,,$$

we have

$$\begin{aligned} \int \phi \, d\mu_a - \int \phi \, d\mu_b &\geq \int_a^b \int \big(\phi H^2 - \langle \nabla \phi, \mathbf{H} \rangle \big) \, d\mu_t \, dt \\ &\geq \int_a^b \int \left(\frac{1}{2} \phi |\mathbf{H}|^2 - \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} \right) \, d\mu_t \, dt \end{aligned}$$

Rearranging, we obtain

$$\begin{aligned} \int_a^b \int \frac{1}{2} \phi |\mathbf{H}|^2 \, d\mu_t \, dt &\leq \int \phi \, d\mu_a - \int \phi \, d\mu_b + \frac{1}{2} \int_a^b \int \frac{|\nabla \phi|^2}{\phi} \, d\mu_t \, dt \\ &\leq \int \phi \, d\mu_a - \int \phi \, d\mu_b + C(\phi) \int_a^b \int \chi_{\{\phi \neq 0\}} \, d\mu_t \, dt \\ &\leq C(\phi) c_K (1+b-a) \,, \end{aligned}$$

where $\{\phi \neq 0\} \subset K$. □

Theorem 5.6. *An integral n-dimensional Brakke flow satisfies*

$$\lim_{\tau \nearrow t} \mu_\tau \geq \mu_t \geq \lim_{\tau \searrow t} \mu_\tau \,.$$

In other worhds, for $\phi \in C_c^0(\mathbb{R}^{n+1})$ with $\phi \geq 0$, we have

$$\lim_{\tau \nearrow t} \int \phi \, d\mu_\tau \geq \int \phi \, d\mu_t \geq \lim_{\tau \searrow t} \int \phi \, d\mu_\tau \,.$$

Proof. First, assume that $\phi \in C_c^2(\mathbb{R}^{n+1})$ with $\phi \geq 0$ (the general case follows by approximation). Then, we have

$$\begin{aligned}\int \phi\, d\mu_d - \int \phi\, d\mu_c &\leq \int_c^d \int \left(-\phi|\mathbf{H}|^2 + \langle \mathbf{H}, \nabla\phi\rangle\right)\, d\mu_t\, dt \\ &\leq \int_c^d \int \frac{1}{2}\frac{|\nabla\phi|^2}{\phi}\, d\mu_t\, dt \\ &\leq C(\phi) c_{\operatorname{supp}\phi}(d-c)\,.\end{aligned}$$

Thus

$$f(t) := \int \phi\, d\mu_t - C(\phi) c_{\operatorname{supp}\phi} t$$

is decreasing in t. This implies that

$$f(t^-) \geq f(t) \geq f(t^+)\,,$$

which finishes the proof, as the linear part of f is continuous. □

Note that we have shown

Theorem 5.7. *For an inegral n-dimensional Brakke flow on $\mathbb{R}^{n+1}$ and $\phi \in C_c^2(\mathbb{R}^{n+1}), \phi \geq 0$, the map*

$$t \mapsto \int \phi\, d\mu_t - C(\phi) c_{supp\,\phi} t$$

is decreasing.

5.1 A compactness theorem for integral Brakke flows

Theorem 5.8. *Suppose that $[a,b] \ni t \mapsto \mu_t^i$ is a sequence of integral Brakke flows. Assume that the local bounds on area are uniform, i.e.*

$$\sup_i \sup_{t\in[a,b]} \mu_t^i(K) \leq c_K < \infty$$

for all $K \Subset \mathbb{R}^{n+1}$. Then after passing to a subsequence

(1) we have weak convergence $\mu_t^i \rightharpoonup \mu_t$ for all $t \in [a,b]$,

(2) $[a,b] \ni t \mapsto \mu_t$ is an integral Brakke flow,

(3) for a.e. $t \in [a,b]$, after passing to a further subsequence which depends on t, the associated varifolds converge nicely $V_t^i \to V_t$.

Proof. Choose $\phi \in C_c^2(\mathbb{R}^{n+1}), \phi \geq 0$. Recall that

$$L_i^\phi(t) = \int \phi \, d\mu_t^i - c(\phi) c_{\operatorname{supp}\phi} t$$

is a sequence of uniformly bounded, decreasing functions of t. Passing to a subsequence depending ϕ, we have that $L_i^\phi(t)$ converges pointwise to a decreasing functin $L(t)$. Hence,

$$\int \phi \, d\mu_t^i$$

has a limit, for all $t \in [a,b]$. Now choose a countable, dense subset $\mathcal{S} \subset C_c^0(\mathbb{R}^{n+1}; \mathbb{R}^+)$ of functions in $C_c^2(\mathbb{R}^{n+1}; \mathbb{R}^+)$. Repeating the above process for each $\phi \in \mathcal{S}$ (choosing a diagonal subsequence), we see that there is a subsequence in i such that for all $\phi \in \mathcal{S}$,

$$\int \phi \, d\mu_t^i$$

has a limit, for all $t \in [a,b]$. By density this extends to all of $C_c^0(\mathbb{R}^{n+1}; \mathbb{R}^+)$. Since the limits are unique, we have that

$$\mu_t^i \rightharpoonup \mu_t$$

for a family of Radon measures $[a,b] \ni t \mapsto \mu_t$. We now want to show that this is a Brakke flow and prove the stated strengthened convergence.

Now, we replace $\mathbb{R}^{n+1}$ by $U \Subset \mathbb{R}^{n+1}$, for simplicity. Thus we may assume that $\mu_t^i(U) \leq C < \infty$ independent of i and t. Note that we have also shown earlier that we can thus assume that

$$\int_a^b \int |\mathbf{H}|^2 d\mu_t^i \, dt \leq D < \infty$$

independent of i. Let $[c,d] \subset [a,b]$. Then

$$\int \phi \, d\mu_c^i - \int \phi \, d\mu_d^i \geq \int_c^d \int \left(\phi |\mathbf{H}_i|^2 - \langle \nabla \phi, \mathbf{H}_i \rangle - \frac{\partial \phi}{\partial t} \right) d\mu_t^i \, dt \, .$$

Thus,

$$\int \phi \, d\mu_c^i - \int \phi \, d\mu_d^i + \varepsilon D + \int_c^d \int \frac{1}{2\varepsilon}|\nabla\phi|^2 \, d\mu_t^i \, dt$$
$$\geq \int_c^d \int \left(\phi|\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle + \varepsilon|\mathbf{H}_i|^2 + \frac{1}{2\varepsilon}|\nabla\phi|^2 - \frac{\partial\phi}{\partial t} \right) d\mu_t^i \, dt$$

Note that

$$\langle \nabla\phi, \mathbf{H}_i \rangle \leq \frac{1}{2\varepsilon}|\nabla\phi|^2 + \frac{\varepsilon}{2}|\mathbf{H}_i|^2 ,$$

so

$$\phi|\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle + \varepsilon|\mathbf{H}_i|^2 + \frac{1}{2\varepsilon}|\nabla\phi|^2 \geq \frac{\varepsilon}{2}|\mathbf{H}_i|^2 ,$$

which in particular is positive. Now, we may pass to the limit in i and use Fatou's lemma to see that

$$\int \phi \, d\mu_c - \int \phi \, d\mu_d + \varepsilon D + \int_c^d \int \frac{1}{2\varepsilon}|\nabla\phi|^2 \, d\mu_t \, dt$$
$$\geq \int_c^d \liminf_i \int \left(\phi|\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle + \varepsilon|\mathbf{H}_i|^2 + \frac{1}{2\varepsilon}|\nabla\phi|^2 \right) d\mu_t \, dt - \int_c^d \int \frac{\partial\phi}{\partial t} \, d\mu_t \, dt .$$

Thus, for a.e. $t \in [c, d]$ we have

$$C(t) := \liminf_i \int \left(\phi|\mathbf{H}_i|^2 - \langle \nabla\phi, \mathbf{H}_i \rangle + \varepsilon|\mathbf{H}_i|^2 + \frac{1}{2\varepsilon}|\nabla\phi|^2 \right) d\mu_t < \infty .$$

Pass to a subsequence (depending on t!) so that this becomes a limit rather than a $\liminf$. Because the integrand is bounded from below by $\frac{\varepsilon}{2}|\mathbf{H}_i|^2$, we see that the μ_t^i are integral varifolds with mean curvature uniformly in $\mathscr{L}^2(\mu_t^i)$. Hence, we can apply the strengthened form of Allard's compactness theorem to pass to a subsequence so that

$$V_t^i \rightharpoonup V_t ,$$

where V_t is an integral varifold with $\mathbf{H} \in \mathscr{L}^2(d\mu_V)$. In particular

$$\int \langle \mathbf{H}_i, X \rangle \, d\mu_{V_t^i} \to \int \langle \mathbf{H}, X \rangle \, d\mu_{V_t}$$

for $X \in C_c(U; \mathbb{R}^{n+1})$ a continuous vector field.

Note that for a.e. t, $V(t)$ is well defined independent of the subsequence depending on t.

This is because an integral varifold V is uniquely determined by its associate measure μ_V. However, we emphasise that the convergence of V_t^i to V as varifolds requires extracting a subsequence depending on t, as we have done above.

Now, returning to $C(t)$, each term converges to what we expect, except for the $|\mathbf{H}_i|^2$ terms, which might drop in general (by weak convergence (just use duality)). Hence we see that

$$C(t) \geq \int \left(\phi|\mathbf{H}|^2 - \langle\nabla\phi, \mathbf{H}\rangle + \varepsilon|\mathbf{H}|^2 + \frac{1}{2\varepsilon}|\nabla\phi|^2\right) d\mu_t\,.$$

Cancelling the terms with $\frac{1}{2\varepsilon}|\nabla\phi|^2$ and letting $\varepsilon \to 0$, this goes in the right direction to conclude that μ_t is an integral Brakke flow. □

5.1.1 Self shrinkers

By the proof of uniform mass bounds for Brakke flows and theorem 5.7 we also get the follwoing extension result:

Corollary 5.9. *Suppose that $[t_1, t_2) \ni t \mapsto \mu_t$ is an integral m-dimensional Brakke flow on $U \subset \mathbb{R}^N$. Then $\mu_{t_2^-} := \lim_{t \nearrow t_2} \mu_t$ exists.*

From this we obtain:

Corollary 5.10. *Suppose that μ_V is an m-dimensional self-shrinker, i.e. V is integral m-dimensional varifold on $\mathbb{R}^N$, such that μ_t defined via*

$$\mu_t(A) := (-t)^{\frac{m}{2}} \mu_V\left((-t)^{-\frac{1}{2}} A\right)$$

for $A \subset \mathbb{R}^N$, $t < 0$, we have that

$$(-\infty, 0) \ni t \mapsto \mu_t$$

is a Brakke flow, or alternatively, V is stationary for the weighted area $\int e^{-\frac{|x|^2}{4}}\, d\mu_V$. Then

(1) $\sup_{r>0} r^{-m} \mu_V(B_r(0)) < \infty$

(2) *μ_V is asymptotic to a unique cone at infinity in a weak sense.*

Proof. We have seen from Corollary 5.9 that

$$\lim_{t \nearrow 0} \mu_t = \mu$$

exists. Since μ_t is just a rescaling of μ_{-1} and the limit is independent of the sequence $t_i \searrow 0$, we see immediately that μ is a (weak) cone (i.e. the measure is invariant under scalings). Furthermore

$$\limsup_{t \nearrow 0} \frac{\mu_{-1}\big(\bar{B}_{(-t)^{-1/2}}(0)\big)}{(-t)^{m/2}} = \limsup_{t \nearrow 0} \mu_t(\bar{B}_1(0)) \leq \mu(\bar{B}_1(0)) < \infty\,,$$

proving the theorem. □

5.2 Existence by elliptic regularisation

We now describe Ilmanen's construction [22] of Brakke flows by "elliptic regularisation". For the technical details see [22].

Theorem 5.11. *Let z denote the height function in $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$ and $\vec{e}$ the upward pointing unit vector. Then $M \subset \mathbb{R}^{n+2}$ is a critical point for $\int_M e^{-\lambda z}\, dA$ if and only if*

$$t \mapsto M - \lambda t \vec{e}$$

is a mean curvature flow.

Proof. If $s \mapsto M_s$ is a variation of M with velocity X, we compute

$$\frac{d}{ds} \int_{M_s} e^{-\lambda z}\, dA = \int \langle -\mathbf{H} + \nabla^{\perp}(-\lambda z), X \rangle e^{-\lambda z}\, dA = \int \langle -\mathbf{H} - \lambda \vec{e}^{\perp}, X \rangle e^{-\lambda z}\, dA\,.$$

On the other hand, note that the flow $t \mapsto M - \lambda t \vec{e}$ has velocity $-\lambda \vec{e}$ and thus normal velocity $-\lambda \vec{e}^{\perp}$. Comparing these two computations proves the theorem. □

Now, for Σ a compact n-dimensional surface in $\mathbb{R}^{n+1}$, let $M_\lambda \subset \mathbb{R}^{n+2}$ minimize $\int e^{-\lambda z}\, dA$

subject to the constraint $\partial M_\lambda = \Sigma$. We can show that M_λ exists and in nice situations (e.g, for hypersurfaces of low dimensions) is regular except for a small singular set. So by the above computation $t \mapsto M_\lambda - \lambda t\vec{e}$ is a mean curvature flow. (This also works in the general case, it then yields a translating Brakke flow).

Our goal is to send $\lambda \to +\infty$. We would like to show that these converge to a limit Brakke flow μ_t which is translation invariant, i.e. $\mu_t = \Sigma_t \times \mathbb{R}$ for Σ_t an n-dimensional Brakke flow in $\mathbb{R}^{n+1}$ with $\Sigma_0 = \Sigma$.

Set $M_\lambda(a,b) = M_\lambda \cap \{a < z < b\}$ and set $S_\lambda(z_0) = M_\lambda \cap \{z = z_0\}$ (see figure in the handwritten notes). Then, we have that, for ν the upward pointing normal vector to $\partial M_\lambda(a,b)$ in $M_\lambda(a,b)$

$$
\begin{aligned}
0 = \int_{M_\lambda(a,b)} \operatorname{div}_M(\vec{e}) &= \int_{M_\lambda(a,b)} -\langle \mathbf{H}, \vec{e}\rangle + \int_{S_\lambda(b)} \langle \vec{e}, \nu\rangle - \int_{S_\lambda(a)} \langle \vec{e}, \nu\rangle \\
= \int_{M_\lambda(a,b)} \lambda |\vec{e}^\perp|^2 + \int_{S_\lambda(b)} |\vec{e}^T| - \int_{S_\lambda(a)} |\vec{e}^T| \,.
\end{aligned}
$$

We may rearrange this to yield

$$
\int_{M_\lambda(a,b)} \lambda |\vec{e}^\perp|^2 + \int_{S_\lambda(b)} |\vec{e}^T| = \int_{S_\lambda(a)} |\vec{e}^T| \,.
$$

In particular

$$
z \mapsto \int_{S_\lambda(z)} |\vec{e}^T|
$$

is a decreasing function. Now, we have

$$
\begin{aligned}
\operatorname{area}(M_\lambda(a,b)) &= \int_{M_\lambda(a,b)} |\vec{e}^\perp|^2 + |\vec{e}^T|^2 \\
&\le \frac{1}{\lambda} \int_{S_\lambda(0)} |\vec{e}^T| + \int_{M_\lambda(a,b)} |\vec{e}^T|^2 \\
&= \frac{1}{\lambda} \int_{S_\lambda(0)} |\vec{e}^T| + \int_{z=a}^{z=b} \int_{S_\lambda(z)} |\vec{e}^T| \\
&\le (\lambda^{-1} + b - a) \int_{S_\lambda(0)} |\vec{e}^T| \\
&\le (\lambda^{-1} + b - a) \operatorname{area}(\Sigma) \,.
\end{aligned}
$$

Thus, the flows have uniform area bounds on compact sets in space-time. Thus, a subsequence converges to a limit Brakke flow (strictly speaking, these flows have boundary, but we could work in the upper half-space, i.e. $\{z > 0\}$, where they do not have boundary).

We have thus obtianed a Brakke flow $\mathbb{R}^+ \ni t \mapsto \mu_t$ in $\mathbb{R}^{n+1} \times \mathbb{R}^+$. We would like to show that (1) the flow is translation invariant, i.e. $\mu_t = \Sigma_t \times \mathbb{R}^+$ for Σ_t a Brakke flow in $\mathbb{R}^{n+1}$ and (2) the has initial condition $\Sigma \times \mathbb{R}^+$.

We start showing the translation invariance. Suppose ϕ is a nice compactly supported nonnegative function on $\mathbb{R}^{n+1} \times \mathbb{R}^+$. Define $\phi^\tau(x, z) = \phi(x, z - \tau)$ to be "upward translation by τ". Let $t \mapsto \mu_t^\lambda$ be the Brakke flow constructed above, which limits to μ_t along some subsequence $\lambda \to \infty$.

Note that (we will use the shorthand $\nu(f) = \int f \, d\nu$ for ν a Radon measure)

$$\mu_t^\lambda(\phi^\tau) = \mu_{t+\tau/\lambda}^\lambda(\phi) \, .$$

Recall that there is a constant c_ϕ depending on ϕ, but independent of λ so that

$$t \mapsto \mu_t^\lambda(\phi) - c_\phi t$$

is decreasing in time. Hence, if $t < s$, for λ large so that

$$t < t + \tau/\lambda < s \, ,$$

we see that

$$\mu_t^\lambda(\phi) - c_\phi t \geq \mu_{t+\tau/\lambda}^\lambda(\phi) - c_\phi(t + \tau/\lambda) = \mu_t(\phi^\tau) - c_\phi(t + \tau/\lambda) \geq \mu_s^\lambda(\phi) - c_\phi s \, .$$

Sending $\lambda \to \infty$ along the subsequence so that $\{\mu_t^\lambda\}$ converges to $\{\mu_t\}$, we have that

$$\mu_t(\phi) - c_\phi t \geq \mu_t(\phi^\tau) - c_\phi t \geq \mu_s(\phi) - c_\phi s \, ,$$

i.e. we have

$$\mu_t(\phi) \geq \mu_t(\phi^\tau) \geq \mu_s(\phi) - c_\phi(s - t) \, .$$

Sending $s \nearrow t$, we have

$$\mu_t(\phi) \geq \mu_t(\phi^\tau) \geq \mu_{t^+}(\phi) \, .$$

This holds for every t. Moreover, for all but countably many t, μ_t is continuous at t. Thus, we see that for a.e. t, $\mu_t = \Sigma_t \times \mathbb{R}^+$, as desired.

Now, we would to show that $\mu_0 = \mu_{\Sigma\times\mathbb{R}^+}$. To do so, we will use the flat norm $\mathscr{F}(\cdot)$. For A, B closed m-dimensional cycles, the flat norm $\mathscr{F}(A-B)$ is the infimum of the area of $m+1$ chains spanning $A-B$.

Let π denote the projection onto $\mathbb{R}^n \times \{b\}$ and let $A_\lambda = \pi(M_\lambda(0,b))$. We compute by the area formula

$$\operatorname{area}(A_\lambda) \le \int_{M_\lambda(0,b)} |\vec{e}^{\perp}| \le \left(\int_{M_\lambda(0,b)} |\vec{e}^{\perp}|^2\right)^{\frac{1}{2}} (\operatorname{area}(M_\lambda(0,b)))^{\frac{1}{2}} .$$

The area term is uniformly bounded. Moreover, we have seen above that the divergence theorem implies that

$$\int_{M_\lambda(0,b)} |\vec{e}^{\perp}|^2 \le \lambda^{-1}\operatorname{area}(\Sigma) .$$

Putting this together, we see that $\operatorname{area}(A_\lambda) \to 0$. Because b is bounded, we can then use this to see that the area in the blue region in the figure (see the handwritten notes from class) is tending to zero.

This shows that

$$\mathscr{F}(M_\lambda(0,b) + A_\lambda - \Sigma \times [0,b]) \to 0 .$$

Recall that the mass is lower semicontinuous under flat convergence. In particular, we have that

$$\begin{aligned}
\operatorname{area}(\Sigma \times [0,b]) &\le \liminf_{\lambda\to\infty} \operatorname{area}(M_\lambda(0,b) + A_\lambda) \\
&\le \liminf_{\lambda\to\infty} \operatorname{area}(M_\lambda(0,b)) \\
&\le \limsup_{\lambda\to\infty} \operatorname{area}(M_\lambda(0,b)) \\
&\le \limsup_{\lambda\to\infty} (\lambda^{-1} + b) \operatorname{area}(\Sigma) \\
&= b \operatorname{area}(\Sigma) \\
&= \operatorname{area}(\Sigma \times (0,b)) .
\end{aligned}$$

In particular, in addition to flat norm convergence, the masses converge (rather than dropping down)! We may now use the following general result:

Proposition 5.12. *Suppose $T_i \to T$ in flat norm. Then, we have seen that the masses satisfy $\mathbb{M}(T) \leq \liminf \mathbb{M}(T_i)$. By passing to a subsequence, we may assume that the associated Radon measures μ_{T_i} converge. Then, we have that*

$$\mu_T \leq \liminf_{i\to\infty} \mu_{T_i} .$$

Roughly speaking, this means that even locally mass can only drop down (we already used the global version of this fact). So, if the total mass converges, then the measures must converge (if they dropped down somewhere, then because the mass cannot jump up somewhere else, this would mean that the mass actually dropped down).

This combines to show that $\mu_0 = \Sigma \times \mathbb{R}^+$. Thus, we have completed the existence theory. Note that we have found solutions Σ_t with the extra convenient property that the flow $t \mapsto \Sigma_t \times \mathbb{R}^+$ is the limit of smooth (or with a small singular set) flows.

5.2.1 Why doesn't the flow disappear immediately?

A natural worry is that the flow we just constructed immediately disappears (this is a well defined Brakke flow!). We will show that the Brakke flows constructed by elliptic regularization cannot disappear, at least for a short time interval.

Definition 5.13. *The support of a Brakke flow in $U, [0,\infty) \ni t \mapsto \mu_t$ is defined as*

$$\overline{\bigcup_t supp(\mu_t) \times \{t\}} \subset U \times \mathbb{R}^+ .$$

Note that without taking the closure, this is unlikely to be a closed set: for example the shrinking sphere sweeps out a paraboloid in space-time, but then disappears, so without the closure, we would be missing the point where the flow shrinks away.

We'll prove the following fact later using the monotonicity formula:

Fact: *If Brakke flows converge, then so do their supports (in the Hausdorff sense).*

Now fix $\Sigma \subset \mathbb{R}^N$ an initial closed embedded hypersurface. Choose p, q inside and outside of Σ respecitvely. We can find small spheres $S(p), S(q)$ around p, q so that they are disjoint from Σ. When we minimise the $e^{-\lambda z}$-weighted area, then M_λ (the minimiser with $\partial M_\lambda = \Sigma$) will be disjoint from $S(p)_\lambda$ and $S(q)_\lambda$. Moreover, this will remain true as $\lambda \to \infty$. Note that we know explicitly what $S(p)_\lambda$ and $S(q)_\lambda$ converge to because we understand shrinking spheres.

Now take a straight line from p to q. Note that $M_\lambda(t)$ must intersect this line at least until it intersects one of the p, q. But the shrinking spheres show that this is only possible after a definite amount of time, say ε. So, for $0 \le t \le \varepsilon$, $M_\lambda(t)$ will always intersect this line. Thus, the limit flow cannot shrink away immediately!

5.2.2 Ilmanen's enhanced flow

Ilmanen has observed that his elliptic regularization procedure can be further refined to give the space-time track a structure of an integral current (or flat chain). To do so, one can work at the level of the surfaces M_λ and show that their space-time track has the structure of a current/chain. Taking the limit (note that the current/chain could loose mass in the limit), we see that the space-time track admits a current/chain with the same support as the support of the Brakke flow, as defined above. This is a very convenient property, as it allows for the use of homological arguments, e.g., the discussion of (signed) intersection numbers, etc.

5.3 Monotonicity and entropy

In this section we will first establish that Huisken's monotonicity formula also holds for Brakke flows and use it to show that one always has the existence of self-similarly shrinking tangent flows. In the following we will always assume that our initial measure of the Brakke flow has bounded n-dimensional area ratios, that is

$$\sup_{x \in \mathbb{R}^{n+1}} \sup_{r>0} \frac{\mu_0(B_r(x))}{\omega_n r^n} \le D < \infty \tag{5.1}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

Lemma 5.14. *Assume $(\mu_t)_{t\geq 0}$ is an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$ satisfying (5.1). Then for $t \in [0, r^2/(4n)]$*

$$\sup_{x\in\mathbb{R}^{n+1}} \mu_t(B_r(x)) \leq 2^{n+2} D\, r^n\,.$$

Proof. This follows from (5.1) together with Theorem 5.3. Exercise. □

We recall that for $X_0 = (x_0, t_0)$ a point in space time, we consider the (scaled) backwards heat kernel

$$\rho_{X_0}(x,t) := (4\pi(t_0 - t))^{-n/2} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}\,.$$

Proposition 5.15. *Let $(\mu_t)_{t\geq 0}$ be an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$ satisfying (5.1). Then for $X_0 = (x_0, t_0)$ with $t_0 > 0$ we have for all $0 \leq t_1 < t_2 < t_0$,*

$$\int \rho_{X_0}(\cdot, t_2)\, d\mu_{t_2} + \int_{t_1}^{t_2} \int \left|\mathbf{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2 \rho_{X_0}\, d\mu_t\, dt \leq \int \rho_{X_0}(\cdot, t_1)\, d\mu_{t_1}\,.$$

Proof. By a translation in space we can assume $x_0 = 0$. We denote $\rho = \rho_{(0,t_0)}$. We recall that from the definition of Brakke flow, we have

$$\int \phi(\cdot, t_2)\, d\mu_{t_2} - \int \phi(\cdot, t_1)\, d\mu_{t_1} \leq \int_{t_1}^{t_2} \int \left(-|\mathbf{H}|^2\phi + \langle \mathbf{H}, \nabla\phi\rangle + \frac{\partial \phi}{\partial t}\right) d\mu_t\, dt\,,$$

for $\phi \in C^1_c(\mathbb{R}^{n+1}\times\mathbb{R};\mathbb{R}^+)$. Whenever the inner integral is finite (i.e. μ_t comes from an n-dimensional varifold with first variation in $\mathscr{L}^2_{\mathrm{loc}}(\mu_t)$), we can calculate, using the first variation formula

$$\begin{aligned}\int -|\mathbf{H}|^2\phi + \langle \mathbf{H}, \nabla\phi\rangle + \frac{\partial\phi}{\partial t}\, d\mu_t &= \int -|\mathbf{H}|^2\phi + 2\langle \mathbf{H}, \nabla\phi\rangle + \mathrm{div}_{\mathrm{Tan}(V_{\mu_t})}(\nabla\phi) + \frac{\partial\phi}{\partial t}\, d\mu_t\\ &= \int -\phi\left|\mathbf{H} - \frac{\nabla^\perp\phi}{\phi}\right|^2 + Q_{\mathrm{Tan}(V_{\mu_t})}(\phi)\, d\mu_t\end{aligned}$$

where for any n-dimensional subspace T of $\mathbb{R}^{n+1}$

$$Q_T(\phi) = \frac{|\nabla^\perp\phi|^2}{\phi} + \mathrm{div}_T(\nabla\phi) + \frac{\partial\phi}{\partial t}\,.$$

Note that $Q_T(\rho) = 0$.

To insert ρ into the above formula, let $\psi = \psi_R$ be a cutoff function with $\chi_{B_R(0)} \leq \psi \leq \chi_{B_{2R}(0)}$, $R|\nabla\psi| + R^2|\nabla^2\psi| \leq C$. We calculate

$$Q_T(\psi\rho) = \psi\, Q_T(\rho) + \rho\, Q_T(\psi) + 2\langle\nabla\psi, \nabla\rho\rangle \leq C\left(\frac{1}{R^2} + \frac{1}{t_0 - t}\right)\chi_{B_{2R}(0)\setminus B_R(0)}\, \rho\,,$$

where we used the fact that $|\nabla\rho| \leq \rho|x|/(2(t_0 - t))$. Inserting $\psi\rho$ above, we obtain

$$\begin{aligned}\int \psi\rho\, d\mu_{t_2} &+ \int_{t_1}^{t_2}\int \psi\left|\mathbf{H} + \frac{x^\perp}{2(t_0 - t)} - \frac{\nabla^\perp\psi}{\psi}\right|^2 \rho\, d\mu_t\, dt \\ &\leq \int \psi\rho\, d\mu_{t_1} + \left(\frac{C}{R^2} + \frac{C}{t_0 - t_2}\right)\int_{t_1}^{t_2}\int_{B_{2R}(0)\setminus B_R(0)} \rho\, d\mu_t\, dt\end{aligned}$$

Note that Lemma 5.14 implies that

$$\sup_{t_1 \leq t \leq t_2} \int \rho\, d\mu_t < \infty\,.$$

Thus the result follows by the monotone and dominated convergence theorems. □

5.3.1 Entropy

For $M^n \subset \mathbb{R}^{n+1}$, we define

$$F(M) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_M e^{-\frac{|x|^2}{4}}\, d\mathcal{H}^n\,,$$

or more generally for a Radon measure μ, we set

$$F_n(\mu) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_M e^{-\frac{|x|^2}{4}}\, d\mu\,.$$

We set $A(r) = \mu(B(0, r))$. Note that $A(r)$ is increasing in r and thus $A'(r)$ exists as a radon measure on $\mathbb{R}$. Then we have by integration by parts (assuming that $A(r)$ grows sub-exponentially) that

$$F_n(\mu) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_0^\infty e^{-\frac{r^2}{4}} A'(r)\, dr = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_0^\infty \frac{\omega_n}{2} r^{n+1} e^{-\frac{r^2}{4}}\left(\frac{A(r)}{\omega_n r^n}\right) dr\,.$$

Note that we can also estimate for any $r_0 > 0$

$$\begin{aligned} F_n(\mu) \geq & \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{r_0}^{\infty} \frac{r}{2} e^{-\frac{r^2}{4}} A(r)\, dr = -\frac{1}{(4\pi)^{\frac{n}{2}}} \int_{r_0}^{\infty} A(r) \frac{d}{dr} \left(e^{-\frac{r^2}{4}} \right) dr \\ & = \frac{1}{(4\pi)^{\frac{n}{2}}} A(r_0) e^{-\frac{r_0^2}{4}} \end{aligned}$$

In particular we see that $F_n(\mu)$ controls the area ratios and is controlled by their supremum, i.e.

$$\frac{\omega_n r^n}{(4\pi)^{\frac{n}{2}}} e^{-\frac{r^2}{4}} \left(\frac{A(r)}{\omega_n r^n} \right) \leq F_n(\mu) \leq C \sup_{r \geq 1} \frac{A(r)}{\omega_n r^n} \,.$$

Colding and Minicozzi have defined [9] a related quantity, *entropy*, as

$$\lambda(M) = \sup_{\lambda > 0, p \in \mathbb{R}^{n+1}} F(\lambda M + p) \,,$$

where we define $\lambda_n(\mu)$ correspondingly. By the above bounds, we see that there exists $C = C(n) > 0$ such that for $A(p,r) = \mu(B_r(p))$,

$$C^{-1} \sup_{\substack{p \in \mathbb{R}^{n+1}, \\ r > 0}} \frac{A(p,r)}{\omega_n r^n} \leq \lambda_n(\mu) \leq C \sup_{\substack{p \in \mathbb{R}^{n+1}, \\ r > 0}} \frac{A(p,r)}{\omega_n r^n} \,. \tag{5.2}$$

From the monotonicity formula for Brakke flows we obtain:

Corollary 5.16. *Let $(\mu_t)_{t \geq 0}$ be an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$ satisfying (5.1). Then the entropy $\lambda_n(\mu_t)$ is finite and decreasing with respect to t. Furthermore, the area ratios area uniformly controlled for all time.*

Note that, in comparison, Lemma 5.14 gives control on the area ratios for small $r > 0$ only for short time (with $t \to 0$ as $r \to 0$). The monotonicity formula allows to rule out measure concentration for all times.

5.3.2 Tangent flows

For a given m-dimensional, integral Brakke flow $(\mu_t)_{t\in I}$ on $\mathbb{R}^{n+1}$ and $\lambda > 0$, we denote the parabolically rescaled measures at a point $X_0 = (x_0, t_0)$ by

$$\mu_t^{X_0,\lambda}(A) = \lambda^n \mu_{t_0+\lambda^{-2}t}(\lambda^{-1}\cdot A + x_0) \tag{5.3}$$

for $t \in I_{\lambda,t_0} := \lambda^2(I - t_0)$. It is easy to check that $I_{\lambda,t_0} \ni t \mapsto \mu_t^{X_0,\lambda}$ is again an n-dimensional, integral Brakke flow on $\mathbb{R}^{n+1}$. Furthermore, if the initial flow has entropy bounded by C, then so does $(\mu_t^{X_0,\lambda})$.

Proposition 5.17 (Existence of tangent flows). *For a given n-dimensional, integral Brakke flow $(\mu_t)_{t\in[0,\infty}$ on $\mathbb{R}^{n+1}$ satisfying (5.1), and any point $X_0 = (x_0, t_0)$ with $t_0 > 0$ and any sequence $\lambda_i \to \infty$, there exists a subsequence (labelled the same) and a Brakke flow $(\nu_t)_{t\in\mathbb{R}}$, such that $(\mu_t^{X_0,\lambda_i}) \rightharpoonup (\nu_t)$ (with convergence as guaranteed by the compactness theorem for Brakke flows), and*

$$\nu_t(A) = \nu_t^\lambda(A) := \lambda^n \nu_{\lambda^{-2}t}(\lambda^{-1}\cdot A), \qquad t < 0 \tag{5.4}$$

for all $\lambda > 0$, and ν_{-1} satisfies

$$H + \frac{x^\perp}{2} = 0 \qquad \nu_{-1}\text{-a.e. } x\,. \tag{5.5}$$

Furthermore

$$\int \rho_{(0,0)}(\cdot,t)\,d\nu_t = \lim_{t'\nearrow t_0} \int \rho_{X_0}\,d\mu_{t'} \qquad t < 0\,. \tag{5.6}$$

Proof. We write $\mu_t^\lambda = \mu_t^{X_0,\lambda}$ and $\rho_{(0,0)} = \rho$. By (5.1) and Corollary 5.16 the flows (μ_t^λ) have bounded area ratios, independent of λ (Exercise). By the compactness theorem for Brakke flows, there exists a subsequence (labelled the same) $\lambda_i \to \infty$ such that $(\mu_t^{\lambda_i}) \rightharpoonup (\nu_t)$ for all $t \in \mathbb{R}$. Since the flows have uniformly bounded area ratios, for every $t < 0$ and $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_i \int_{\mathbb{R}^{n+1}\setminus B_R(0)} \rho\,d\mu_t^{\lambda_i} \le \varepsilon\,.$$

Using a suitable cutoff function, weak convergence $\mu_t^{\lambda_i} \rightharpoonup \nu_t$ implies that

$$\int \rho(\cdot,t)\,d\nu_t = \lim_{i\to\infty}\int \rho(\cdot,t)\,d\mu_t^{\lambda_i} = \lim_{t'\nearrow t_0}\int \rho_{X_0}\,d\mu_{t'} \qquad t<0\,,$$

where the last equality follows by the monotonicity formula. Using the monotonicity for $(\nu_t)_{t\in\mathbb{R}}$ cantered at $(0,0)$, yields that for a.e. $t<0$, ν_t is an n-dimensional integral varifold with $\mathbf{H}\in\mathscr{L}^2_{\mathrm{loc}}(\mu_t)$, and

$$\mathbf{H} + \frac{x^\perp}{-2t} = 0 \qquad \nu_t\text{-a.e. } x\,. \tag{5.7}$$

Next we show self-similarity. Define $\tilde\nu_t(A) := (-t)^{-n/2}\nu_t((-t)^{-1/2}A), t<0$. It suffices to show that $\tilde\nu_t$ is constant in t. Let $\phi\in C^2_c(\mathbb{R}^{n+1};\mathbb{R}^+)$ and $\tilde\phi(x,t) = (-t)^{n/2}\phi((-t)^{1/2}x)$. Note that

$$\frac{\partial\tilde\phi}{\partial t} = -\frac{n}{2t}\tilde\phi - \frac{1}{2t}\langle\nabla\tilde\phi, x\rangle\,.$$

By the definition of Brakke flow, we have for $t_1\le t_2<0$

$$\begin{aligned}
\int\phi\,d\tilde\nu_{t_2} - \int\phi\,d\tilde\nu_{t_1} &= \int\tilde\phi\,d\nu_{t_2} - \int\tilde\phi\,d\nu_{t_1}\\
&\le \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde\phi - \tilde\phi|\mathbf{H}|^2 + \langle\nabla\tilde\phi,\mathbf{H}\rangle - \frac{1}{2t}\langle\nabla\tilde\phi,x\rangle\,d\nu_t\,dt\\
&= \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde\phi - \frac{\tilde\phi}{2t}\langle\mathbf{H},x\rangle + \left\langle\nabla\tilde\phi,\frac{x^\perp}{2t}\right\rangle - \frac{1}{2t}\langle\nabla\tilde\phi,x\rangle\,d\nu_t\,dt\\
&= \int_{t_1}^{t_2}\int -\frac{n}{2t}\tilde\phi - \frac{\tilde\phi}{2t}\langle\mathbf{H},x\rangle - \left\langle\nabla\tilde\phi,\frac{x^T}{2t}\right\rangle\,d\nu_t\,dt\,,
\end{aligned}$$

where we used (5.7). From the first variation formula we have

$$\int -\frac{\tilde\phi}{2t}\langle\mathbf{H},x\rangle\,d\nu_t = \int\frac{1}{2t}\mathrm{div}_{T\nu_t}(\tilde\phi\,x)\,d\nu_t = \int\frac{n}{2t}\tilde\phi + \left\langle\nabla\tilde\phi,\frac{x^T}{2t}\right\rangle\,d\nu_t\,.$$

Combining with the above we see that $\int\phi\,d\tilde\nu_t$ is non-increasing in t.

Next assume without loss of generality that $\phi<\rho$ and apply the same calculation to $\rho-\phi$ (using the exponential decay of ρ and the bounded area ratios to validate the insertion of this function) to see that $\int\rho-\phi\,d\tilde\nu_t$ is also non-increasing in t.

It follows by (5.6) that $\int\phi\,d\tilde\nu_t$ is constant in t, which implies (5.4). Thus by (5.7) it follows that (5.5) holds. $\square$

5.3.3 Gaussian density

Define

$$\mathscr{C}_\Lambda = \{\text{Radon measures on } \mathbb{R}^N \text{with } \lambda_m(\mu) \le \Lambda\}$$

Note that this class is preserved by the Brakke flow. Moreover the set of Brakke flows in $\mathscr{C}_\Lambda$ is compact because the entropy bound yields area ratio bounds.

Suppose that f is continuous, bounded (or more generally $|f| \le c(1+|x|)^k$) and $\mu_i \in \mathscr{C}_\Lambda$ has $\mu_i \to \mu \in \mathscr{C}_\Lambda$ (this is always the case up to passing to a subsequence). It is easy to show (using a suitable cut-off function and the area ratio bounds) that

$$\int f e^{-\frac{|x|^2}{4r^2}}\, d\mu_i \to \int f e^{-\frac{|x|^2}{4r^2}}\, d\mu\,.$$

If (μ_t) is a Brakke flow (with space-time track $\mathcal{M}$) and $X = (x_0, t_0), r > 0$, define the *Gaussian density ratios* as

$$\Theta(\mathcal{M}, X, r) = \int \frac{1}{(4\pi r^2)^{\frac{m}{2}}} e^{-\frac{|x-x_0|^2}{4r^2}}\, d\mu_{t_0 - r^2}\,.$$

Note that the monotonicity formula implies that $\Theta(\mathcal{M}, X, r)$ is increasing in r. (Note that this is reminiscent of the monotonicity formula for minimal surfaces).

Hence, as $r \searrow 0$, the limit exists, so we can define that *Gaussian density* of $\mathcal{M}$ at X as

$$\Theta(\mathcal{M}, X) := \lim_{r \searrow 0} \Theta(\mathcal{M}, X, r)\,.$$

Proposition 5.18. *Assume $\mathcal{M}_i, \mathcal{M} \in \mathscr{C}_\Lambda$ and $\mathcal{M}_i \rightharpoonup \mathcal{M}$, $X_i \to X$ then*

$$\Theta(\mathcal{M}, X) \ge \limsup_i \Theta(\mathcal{M}_i, X_i)\,.$$

Proof. Translating by $X - X_i$, we can assume that $X_i = X$. We have that

$$\Theta(\mathcal{M}_i, X) \le \Theta(\mathcal{M}_i, X, r) \to \Theta(\mathcal{M}, X, r)$$

for any $r > 0$. Letting $r \searrow 0$, the proposition follows. $\square$

Proposition 5.19. *Assume $\mathcal{M}_i, \mathcal{M} \in \mathscr{C}_\Lambda$ and $\mathcal{M}_i \rightharpoonup \mathcal{M}$, $X_i \to X$, $r_i \searrow 0$, then*

$$\limsup_i \Theta(\mathcal{M}_i, X_i, r_i) \leq \Theta(\mathcal{M}, X) .$$

Proof. Translating by $X - X_i$, we can assume that $X_i = X$. Then, for $r > 0$, for i sufficiently large, we have that $r_i < r$. Thus

$$\limsup_i \Theta(\mathcal{M}_i, X, r_i) \leq \limsup_i \Theta(\mathcal{M}_i, X, r) = \Theta(\mathcal{M}, X, r) .$$

This holds for all $r > 0$. Letting $r \searrow 0$, the proposition follows. $\square$

We denote parabolic scaling in space-time with a factor $\lambda > 0$ by $\mathcal{D}_\lambda : (x, t) \mapsto (\lambda x, \lambda^2 t)$. For the spacetime track $\mathcal{M}$ of a Brakke flow, $\mathcal{D}_\lambda(\mathcal{M} - X_0)$ is just the parabolic scaling of the flow as defined in (5.3). The previous proposition directly implies:

Theorem 5.20. *Assume $\mathcal{M}_i, \mathcal{M} \in \mathscr{C}_\Lambda$ and $\mathcal{M}_i \rightharpoonup \mathcal{M}$, $X_i \to X$, $\lambda_i \to \infty$, where X_i, X are always strictly after the initial time of the respective flows, then up to a subsequence*

$$\mathcal{D}_{\lambda_i}(\mathcal{M}_i - X_i)$$

converges to an eternal limit Brakke flow $\tilde{\mathcal{M}}$. Moreover $\lambda(\tilde{M}) \leq \Theta(\mathcal{M}, X)$.

Remark 5.21: Recall that we have shown, using the monotonicity formula, that for a Brakke flow $\mathcal{M}$, with X after the initial time, $\lambda_i \to \infty$, that up to a subsequence $\mathcal{D}_{\lambda_i}(\mathcal{M} - X)$ converges to a self-similar tangent flow at X. But the tangent flow does not capture the full behaviour of the flow around the singularity (think of, for example, the degenerate neckpinch or for immersed curves, where a loop 'pinches' off). To capture this behaviour, it is often helpful to consider suitably chosen points $X_i \to X$ and suitable scaling factors $\lambda_i \to \infty$ (such that the limit is non-empty and non-trivial), and consider a subsequential limit of

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - X_i) .$$

Such a limit is called a limit flow at X. Note that it is not necessarily anymore self-similar. But the above theorem shows that the limit flow has entropy bounded by the

Gaussian density at X. This for example has been crucially used in the recent proof of the mean convex neighborhood conjecture by Choi-Haslhofer-Hershkovits [6].

Remark 5.22: Note that even if the flow is only defined on an open subset $U \subset \mathbb{R}^N$, we can define

$$f_R(x,t) = R^{-2}(R^2 - |x|^2 - 2mt)^+$$

and $f_{R,X_0} := f(x - x_0, t - t_0)$. For $x \in U$ and $t_0 > 0$ we can choose R sufficiently small such that the support of $f_{R,X}$ is contained in U for t close to t_0. It can easily be checked that $t \mapsto \mu_t(f^3_{R,X})$ is decreasing on that interval. Even more, one can also show that $t \mapsto \mu_t(f^3_{R,X}\rho_X)$ is decreasing on that interval as well. This allows us to define

$$\Theta(\mathcal{M}, X) = \lim_{r\to 0} \mu_{t-r^2}(f^3_{R,X}\rho_X)\,.$$

This agrees with the other definition of Gaussian density, if μ_t is defined on all of $\mathbb{R}^N$. Recall that we denoted the so localised Gaussian density ratios in Section **??** by $\Theta^R(\mathcal{M}, X, r)$, see Exercise **??**.

Now, suppose that $\mathcal{M}$ is an ancient flow in $\mathbb{R}^N$ in $\mathscr{C}_\Lambda$. Then we may set

$$\Theta(\mathcal{M}) = \lim_{r\to\infty} \Theta(\mathcal{M}, X, r)\,.$$

The limit exists by the monotonicity formula, and is finite since $\mathcal{M}$ is in $\mathscr{C}_\Lambda$. It is a simple exercise to check that $\Theta(\mathcal{M})$ does not depend on X. Furthermore, for every sequence $\lambda_i \to \infty$, up to a subsequence

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - X)$$

converges by the monotonicity formula to self-similarly shrinking flow with Gaussian density ratios equal to $\Theta(\mathcal{M})$. This is called a *tangent flow at infinity.* Note further that

$$\Theta(\mathcal{M}) = \sup_{X, r>0} \Theta(\mathcal{M}, X, r) = \lambda(\mathcal{M})\ ,$$

where $\lambda(\mathcal{M})$ is the supremum in time of the entropy of μ_t.

Recall that if we set

$$\Theta_{\text{euc}}(M, 0, r) = \frac{A(r)}{\omega_m r^m}\,,$$

then the Gaussian area satisfies

$$F(M) \geq \inf_{0<r<\infty} \Theta_{\text{euc}}(M, 0, r) \, .$$

Moreover,

$$F(M) \geq \inf_{\varepsilon<r<\varepsilon^{-1}} \Theta_{\text{euc}}(M, 0, r) - \delta(\varepsilon) \, .$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Corollary 5.23. *We have that*

$$\lambda(M) \geq \Theta_{euc}(M, 0) = \lim_{r \searrow 0} \Theta_{euc}(M, 0, r) \, ,$$

assuming the limit exists. Similarly

$$\lambda(M) \geq \Theta_{euc}(M, \infty) = \lim_{r \to \infty} \Theta_{euc}(M, 0, r) \, ,$$

assuming the limit exists.

Proposition 5.24. *For any non-zero ancient flow, $\Theta(\mathcal{M}) \geq 1$.*

Proof. Let T denote the extinction time of the flow (which could be $+\infty$). For a.e. $t < T, \mu_t$ is a non zero integral varifold. Thus $\Theta_{\text{eucl}}(\mu_t, x)$ exists and is a nonzero integer for μ_t-a.e. x. Thus, for such an x,

$$\lambda(\mu_t) \geq \Theta_{\text{eucl}}(\mu_t, x) \geq 1 \, .$$

Because $\Theta(\mathcal{M})$ is the supremum of $\lambda(\mu_t)$, the claim follows. □

Lemma 5.25. *If $\mathcal{M}$ is a Brakke flow and $X = (x,t)$ is such that $\Theta_{euc}(\mu_t, x)$ exists, then*

$$\Theta(\mathcal{M}, X) \geq \Theta_{euc}(\mu_t, x) \, .$$

Proof. Translate such that $X = (0,0)$. We want to show that

$$\Theta(\mathcal{M}, (0,0)) \geq \Theta_{\text{euc}}(\mu_0, 0) \, .$$

To see this, note that monotonicity implies that for $s > 0$ and some $\delta(r) \to 0$ as $r \to 0$, independent of s

$$\Theta(\mathcal{M},(0,r^2),r+s) \geq \Theta(\mathcal{M},(0,r^2)) \geq \Theta_{\mathrm{euc}}(\mu_0,0)-\delta(r)\,.$$

Sending $r \searrow 0$, we have

$$\Theta(\mathcal{M},(0,0),s) \geq \Theta_{\mathrm{euc}}(\mu_0,0)\,.$$

Now, the claim follows after letting $s \to 0$. □

Suppose we have a sequence of converging Brakke flows $\mathcal{M}_i \rightharpoonup \mathcal{M}$ in $\mathscr{C}_\Lambda$. We have seen that

$$\Theta(\mathcal{M},X) \geq \limsup_i \Theta(\mathcal{M}_i,X_i)$$

if $X_i \to X$.

Corollary 5.26. *If $\mathcal{M}$ is a Brakke flow in $\mathscr{C}_\Lambda$, then $\Theta(\mathcal{M},X) \geq 1$ for all $X \in \operatorname{supp}\mathcal{M}$.*

Proof. We can choose $X_i = (x_i,t_i)$ converging to X such that $\Theta_{\mathrm{euc}}(\mu_{t_i},x_i) \geq 1$. Then the statement follows from the previous lemma. □

5.4 A version of Brakke's regularity theorem and unit regular flows

First, we recall Allard's regularity theorem [1] (see also [24]), which says:

Theorem 5.27 (Allard's regularity theorem)**.** *There is $\varepsilon = \varepsilon(m,N)$ with the following property. If M is a stationary integral varifold and $\Theta_{eucl}(M,x) < 1+\varepsilon$, then x is a regular point of M.*

Since this is a local statement, this result also extends to stationary integral varifolds in an ambient Riemannian manifold (N^N,g). The corresponding regularity theorem for Brakke flows was proven by Brakke [3] (see also [23]).

Theorem 5.28 (Brakke's regularity theorem)**.** *There is $\varepsilon = \varepsilon(m,N)$ with the following property. If $\mathcal{M}$ is an integral Brakke flow and $\Theta(\mathcal{M},X) < 1+\varepsilon$, then X is a regular point of for the flow.*

There is a subtle point about the second statement, because Brakke flows are allowed to suddenly vanish. So, one way to interpret the statement is that in a *backwards* parabolic neigbhorhood of X, the flow is a smooth flow of surfaces. But as we will see later, if the flow comes from elliptic regularization, then it can be seen to be a smooth flow of surfaces both forward and backwards in time.

We recall White's local regularity theorem, Theorem **??**, where we consider parabolic backwards cylinders $P((x_0,t_0),r) = B(x_0,r)\times(t_0-r^2,t_0]$.

Theorem 5.29 (White's local regularity theorem [27])**.** *There exists universal constants $\varepsilon>0$ and $C<\infty$ with the following property: If $\mathcal{M}$ is a smooth mean curvature flow in $P(X_0,4nR)$ such that*

$$\sup_{X\in P(X_0,r)} \Theta^R(\mathcal{M},X,r) < 1+\varepsilon$$

for some $r\in(0,R)$, then

$$\sup_{P(X_0,r/2)} |A| \leq Cr^{-1}\,. \tag{5.8}$$

We can use Allard's theorem and White's local regularity to prove a local gap theorem.

Theorem 5.30 (Gap Theorem)**.** *Suppose that $\mathcal{M}$ is a self-similar Brakke flow which is not a multiplicity 1 plane. Then $\Theta(M)\geq\eta>1$, where $\eta=\eta(m,N)$.*

Proof. Suppose not. Then, there is a sequence of self-similar, non-planar, Brakke flows with $\Theta(\mathcal{M}_i)\to 1$. Note that for i sufficently large we have $\Theta(\mathcal{M}_i)<1+\varepsilon$, where $\varepsilon=\varepsilon(m,N)$ is from Allard's theorem. But then we have for any point $X=(x,-1)$ in the support of $\mathcal{M}_i(-1)$ that $\Theta(\mathcal{M}_i,X)<1+\varepsilon$. But thus by Lemma 5.25 we have that $\Theta_{\mathrm{euc}}(\mathcal{M}_i(-1),x)<1+\varepsilon$ and Allard's theorem yields that $\mathcal{M}_i(-1)$ is smooth and thus $\mathcal{M}_i$ is smooth for $t<0$.

Assuming that ε smaller as well than the constant of White's regularity theorem, we see that the Gaussian density ratios are everywhere controlled by $1+\varepsilon$. However, as the $\mathcal{M}_i$ are non-flat, since they are self-similar, the curvature has to blow up near the origin in space-time. But this contradicts White's local regularity theorem. ∎

Definition 5.31. *We say an integral Brakke flow is* unit-regular *if every point with Gaussian density one has a space-time neighborhood where it is smooth. We denote the class of unit regular Brakke flows by $\mathscr{G}$.*

Theorem 5.32 (Easy Brakke)**.** *The class $\mathscr{G}$ is closed under weak convergence of Brakke flows. Moreover, if $\mathcal{M} \in \mathscr{G}$ and if $\Theta(\mathcal{M}, X) < \eta$, with η from the Gap Theorem, then X is a regular point.*

Note that Brakke's (hard) theorem says that this is true for the set of all integral Brakke flows, but just considering *backwards* parabolic neighborhoods.

Proof. Let $\mathcal{M}_i \in \mathscr{G}$ have $\mathcal{M}_i \rightharpoonup \mathcal{M}$. Suppose that $X \in \mathcal{M}$ and $\Theta(\mathcal{M}, X) \leq \eta - 2\varepsilon$ for some $\varepsilon > 0$. By upper semi-continuity of the density, there is $I > 0$ and some (space-time) neighborhood $\mathcal{U}$ of X so that

$$\Theta(\mathcal{M}_i, \cdot) \leq \eta - \varepsilon$$

in $\mathcal{U}$ for $i \geq I$. Thus, we see that any tangent flow to $\mathcal{M}_i$ at $Y \in \mathcal{U}$ has entropy at most $\eta - \varepsilon$. Thus by the Gap Theorem it has to be a mulitplicity one plane, so $\Theta(\mathcal{M}_i, Y) = 1$. Thus, by unit regularity the flows $\mathcal{M}_i \cap \mathcal{U}$ are smooth mean curvature flows (and no sudden vanishing occurs).

Furthermore, shrinking $\mathcal{U}$ if necessary, we can assume that there is $r > 0$ such that

$$\Theta(\mathcal{M}_i, \cdot, r) \leq \eta - \varepsilon$$

on $\mathcal{M}_i \cap \mathcal{U}$ for all $i \geq I$. Thus by White's local regularity theorem we have $\mathcal{M}_i \to \mathcal{M}$ smoothly on $\mathcal{U}$. $\square$

Lemma 5.33. *Let $\mathcal{M}$ be a translating mean curvature flow on $\mathcal{U}$. Then $\mathcal{M}$ is unit regular.*

Proof. Assume for $X = (x_0, t_0) \in \mathcal{U}$ we have $\Theta(\mathcal{M}, X) = 1$. By Lemma 5.25 we have $\Theta(\mathcal{M}(t_0), x_0) = 1$. Thus Allard's theorem implies that $\mathcal{M}(t_0)$ is smooth in a neighborhood of x_0. Since the flow is translating, there is a full space-time neighborhood of X where $\mathcal{M}$ is smooth. $\square$

Corollary 5.34. *Any Brakke flow constructed via elliptic regularisation is unit regular.*

5.5 Stratification

We would like to discuss the stratification for Brakke flows, see [26] and also [2, 16]. For simplicity we start with the stratification of minimal surfaces. We have the following basic result.

Lemma 5.35. *For $M \subset \mathbb{R}^N$, $0 \in M$, let $V(M) := \{x \in \mathbb{R}^N : M = M + x\}$.*

(1) Trivially, $V(M)$ is an additive subgroup of $\mathbb{R}^N$.

(2) If M is a cone, then $V(M)$ is a linear subspace and $M = C \times V(M)$, where C is the cone $C = M \cap (V(M))^\perp$.

If we assume that M is minimal we can relate $V(M)$ with the top density points.

Theorem 5.36. *If $M \subset \mathbb{R}^N$ is a minimal cone (i.e. a stationary integral varifold which is a cone), then*

$$\max_M \Theta(M, \cdot) = \Theta(M, 0) = \Theta(M)\,.$$

Moreover, if

$$spine(M) = \{x : \Theta(M, x) = \Theta(M)\}\,,$$

then $spine(M) = V(M)$.

Proof. First, from the monotonicity formula we note that $\Theta(M, x) \leq \Theta(M)$, with equality if and only if M is dilation invariant around x.

Now, suppose that $x \in V(M)$. Because $M + x = M$, we see that $\Theta(M, x) = \Theta(M, 0)$. Thus, $x \in \mathrm{spine}(M)$. This shows that $V(M) \subset \mathrm{spine}(M)$.

On the other hand, if $y \in \mathrm{spine}(M)$, then M is invariant by dilation around y and 0. In particula, composing the maps $x \mapsto y + \lambda(x - y) = (1 - \lambda)y + \lambda x$ and $x \mapsto \frac{1}{\lambda}x$, we see that M is invariant under

$$x \mapsto \left(\frac{1-\lambda}{\lambda}\right) y + x\,.$$

Setting $\lambda = \frac{1}{2}$ we see that M is invariant under $x \mapsto x+y$. This shows that spine$(M) \subset V(M)$. $\square$

Now, if M is a stationary integral varifold, we set

$$\Sigma_k = \{x \in M : \text{each tangent cone at } x \text{ has a spine of dimension} \leq k\}.$$

Then, the main result concerning stratification is

Theorem 5.37. *The set* Σ_k *satisfies* $\dim_{Haus}(\Sigma_k) \leq k$.

Example 5.38 (Federer [16])**:** Let us consider M which minimizes m-dimensional area mod 2 (i.e. we work with flat chains mod 2). Clearly, there can be no tangent planes of multiplicity bigger than 1. Thus, every point in $M \setminus \Sigma_{m-1}$ must be regular, so we automatically get $\dim_{Haus}\mathcal{S} \leq m-1$, where $\mathcal{S}$ is the singular set. Next, if we consider $1-$dimensional area minimizing mod 2 cones, we see that they must be the union of rays. Moreover, in order for them to have no boundary mod 2 at the origin, there must be an even number of rays.

Thus, we see that there are no-nontrivial $1-$dimensional cones which minimize area mod 2. From this, we see that any point in $M \setminus \Sigma_{m-2}$ must be a regular point. Putting this together with the stratification theorem, we see that the singular set satisfies $\dim_{Haus}\mathcal{S} \leq m-2$.

For Brakke flows, the situation is similar but slightly more complicated. Let $\mathcal{M}$ be a Brakke flow. We know that

$$\mathcal{D}_\lambda(\mathcal{M} - X) \rightharpoonup \mathcal{M}'$$

subsequentially, where $\mathcal{M}'$ is a *tangent flow at* X. From the monotonicity formula we obtain:

Theorem 5.39. *We have that* $\Theta(\mathcal{M}, X) = \Theta(\mathcal{M}', 0) = \Theta(\mathcal{M}')$*, so* $\mathcal{M}' \cap \{t < 0\}$ *is self-similar, i.e. invariant under parabolic dilations:* $\mathcal{D}_\lambda(\mathcal{M}' \cap \{t < 0\}) = \mathcal{M}' \cap \{t < 0\}$ *for all* $\lambda > 0$.

Observe that we do not have any information about $t \geq 0$! Indeed, below we will see various examples of different possible behaviours for $t \geq 0$.

In general, for $\mathcal{M}$ a self-similar flow, we define the "spatial spine"

$$V(\mathcal{M}) = \{x \in \mathbb{R}^N : \Theta(\mathcal{M}, (x, 0)) = \Theta(\mathcal{M})\}\,.$$

As before it is not difficult to show that (exercise)

$$V(\mathcal{M})\{x \in \mathbb{R}^N : \mathcal{M} \cap \{t < 0\} \text{ is invariant under translation by } (x, 0)\}$$

and that $V(\mathcal{M})$ is a linear subspace of $\mathbb{R}^N$. Moreover, we can show that the set

$$\{X : \Theta(\mathcal{M}, X) = \Theta(\mathcal{M})\}$$

must be one of the following

(1) $V(\mathcal{M}) \times \{0\}$, e.g., a cylinder,

(2) $V(\mathcal{M}) \times \mathbb{R}$, e.g., a minimal cone or a static plane (of possibly higher multiplicity),

(3) $V(\mathcal{M}) \times (-\infty, a]$ for $a \geq 0$, e.g., $\mathcal{M}$ remains a minimal cone until time a, and then flows in some other manner (or vanishes). We call this case a *quasi-static cone*. Similarly we have *quasi-static planes*, which vanish for $t > a$. Note that unit regularity rules out quasi-static multiplicity one planes.

An interesting example of (3) is the tangent flow to a cusp singularity from an immersed plane curve. For $t < 0$, it is a multiplicity 2 line, while for $t > 0$ it is empty. We see the same picture as the blow-down, i.e. the tangent flow at infinity, for a translating grim reaper.

Now, to discuss the stratification of a general Brakke flow, if $\mathcal{M}'$ is a tangent flow, then we set $d(\mathcal{M}')$ to be the dimension of the spatial spine $V(\mathcal{M}')$. Then, we set $D(\mathcal{M}') = d + 2$ if $\mathcal{M}'$ is a static cone for all time, and $D(\mathcal{M}') = d$ otherwise. Note that in case (2) above, $D = d + 2$ and otherwise $D = d$.

Theorem 5.40 (White, [26]). *For $\mathcal{M}$ an integral Brakke flow, let*

$$\Sigma_k = \{X : D(\mathcal{M}') \leq k \textit{ for every tangent flow at } X\}\,.$$

Then

$$\dim_{par\ Haus}(\Sigma_k) \leq k\,.$$

Here, the parabolic Hausdorff dimension is the dimension with respect to the metric on space-time given by, e.g.,

$$d\big((x,t),(x',t,)\big) = |x - x'| + |t - t'|^{\frac{1}{2}}\,.$$

The different scaling in time implies (exercise, compare to usual Fubini where one only looses one dimension for a.e. height in a coordinate direction):

Corollary 5.41. *For a.e.* t_0 *one has*

$$\dim_{Haus}(\Sigma_k \cap \{t = t_0\}) \le k - 2\,.$$

We now discuss an example of stratification. Consider the space-time track of a network of curves, consisting of two loops (one bigger than the other) connected by a straight segment, with equal angles at the triple junctions. Note that the time derivative of the area of a bounded domain is given by $-(2 - l/3)\pi$, where l is the number of corners of the domain (i.e. hexagons have constant area). See the picture drawn in the notes in class.

The space-time track drawn in class has the following tangent flows (see the numbering there):

(a) The tangent flow is a static multiplicity one plane, as this is a regular point.
(b) The tangent flow is a shrinking circle.
(c) This is a quasi-static cone. The tangent flow is a static triple junction for $t < 0$, but at $t = 0$ one arc disappears and the other two flow outwards smoothly (one sees an expanding solution for $t > 0$).
(d) This is a static triple junction.
(e) This tangent flow is what one might call *quasi-regular*: it is a multiplicity one line for $t < 0$, but then disappears at $t = 0$.
(f) This is a self-similar shrinker which looks like a spoon. At $t = 0$ it becomes a half-line, and disappears instantly (one can see this has to happen by using scaled grim reapers as barriers).

5.5.1 Ruling out the worst singularities

Now, if $\mathcal{M}$ is an m dimensional integral Brakke flow, we know that for any tangent flow, $d \leq m$, so $D \leq m+2$ (this is the trivial estimate). So the worst thing we could get, from the point of view if stratification is a higher multiplicity plane which does not disappear. Then $D = m+2$. The net possibility is a static cone with an $(m-1)$-dimensional spine, i.e., a union of half-planes. This has $D = m+1$. So, if we can rule these possibilities out, then we can already say that the parabolic Hausdorff dimension of the singular set is at most m.

Note that a shrinking $\mathbb{S}^1 \times \mathbb{R}^{m-1}$ has $D = m-1$. So in general the dimension of the singular set should be at least $m-1$. Note that this implies for example that a 2-dimensional Brakke flow in $\mathbb{R}^3$ with only multiplicity one singularities of the type $\mathbb{S}^2$ and $\mathbb{S}^1 \times \mathbb{R}$ has a singular set of parabolic Hausdorff dimension at most one. But this implies that the flow is smooth for a.e. time.

5.6 An easy parity theorem

We will restrict ourselves to hypersurfaces, but modified versions of these results hold in general, see the paper of White [28]. We have seen above that to control the size of the singular set, after ruling out higher multiplicity (which is still in general an open problem), one needs to rule out static cones consisting of half-planes meeting along an $m-1$ dimensional subspace. The following is a tool to rule out some of these.

Theorem 5.42. *Define $\mathscr{G}_{cyc}$ the class of unit regular m-dimensional integral Brakke flows in $\mathbb{R}^{m+1}$ such that if a closed curve C has $C \cap sing(\mathcal{M}) = \emptyset$ and C intersects $reg(\mathcal{M})$ (the set of regular, multiplicity one points) transversely, then $C \cap \mathcal{M}$ has an even number of elements. Then $\mathscr{G}_{cyc}$ is closed under weak convergence of Brakke flows.*

In the literature flows in $\mathscr{G}_{\mathrm{cyc}}$ are often called unit regular and cyclic mod 2.

Proof. Pick $\mathcal{M}$, limit of flows in $\mathscr{G}_{\mathrm{cyc}}$, and pick such a curve C. As we have seen in the proof that $\mathscr{G}$ is closed under weak convergence, the convergence to $\mathcal{M}$ is smooth in a neighborhood of $\mathrm{reg}(\mathcal{M})$ and thus in a neighborhood of C. Thus, the desired property passes to limit. $\square$

Remark 5.43: Note that in codimension one we can do elliptic regularisation using Caccioppoli sets (sets of finite perimeter). Thus our approximating translators have a well defined inside and outside (especially at every regular point). This implies that the approximating solutions are in $\mathscr{G}_{\mathrm{cyc}}$ and thus we see the constructed limiting Brakke flow is in $\mathscr{G}_{\mathrm{cyc}}$ as well.

Theorem 5.44. *Let $\mathcal{M}$ be an m-dimensional integral Brakke flow in $\mathscr{G}_{cyc}(\mathbb{R}^{m+1})$. Consider the set*

$$W = \{X : \Theta(\mathcal{M}, X) < 2\},$$

which is open by upper semi-continuity if density. Then $sing(\mathcal{M}) \cap W$ has parabolic Hausdorff dimension at most $m-1$. Moreover, away from a set of dimension at most $m-2$, $sing(\mathcal{M}) \cap W$ has tangent flows which are all $C \times \mathbb{R}^{m-3}$ for C a static smooth 3-d cone, or $\mathbb{S}^1 \times \mathbb{R}^{m-1}$.

Remark 5.45: Colding-Ilmanen-Minicozzi [10] have shown that if one tangent flow is $\mathbb{S}^k \times \mathbb{R}^{m-k}$, then they all are. Subsequently Colding-Minicozzi [11] showed that in this case, the tangent flow is unique, i.e., there is no rotation.

Proof of Theorem 5.44. To prove the above theorem, we consider the possible tangent flows at a singularity with density $\Theta < 2$. first we consider the static/quasi-static cones:

(1) The first possibility would be a static plane of multiplicicty ≥ 2. This could contribute dimension $m+2$ to the singular set. But, it cannot happen by density considerations.

(2) Similarly, a quasi-static plane of multiplicity ≥ 2 could contribute dimension m, but it is also ruled out density considerations.

(3) A static, (resp. quasi-static) union of half-planes (i.e., a 1-D minimal cone times $\mathbb{R}^{m-1}$) could contribute $m+1$ (resp. $m-1$). However, $\Theta < 2$ implies that there must be exactly 3 half-planes of multiplicity 1, which is ruled out by parity, or otherwise the cone is a flat, multiplicity one cone.

(4) A static (resp. quasi-static) 2-D minimal cone crossed with $\mathbb{R}^{m-2}$ could contribute dimension m (resp. $m-2$). such a cone intersected with the unit sphere is a

geodesic network. By $\Theta < 2$ and parity considerations, there cannot be any junctions, so such a cone cannot exist (besides a multiplicity one plane).

Thus, we see that the worst static cone could happen is a 3-D cone times $\mathbb{R}^{m-3}$, contributing dimension at most $m-1$. We must also consider the possible shrinkers:

(1) One possibility is a 2-D shrinker times $\mathbb{R}^{m-1}$, which could contribute dimension $m-1$. The argument above shows that the 1-D shrinker cannot have any junctions (e.g., it cannot be the shrinking spoon). Hence, it is a smooth, embedded shrinker, and is this a round $\mathbb{S}^1$. Thus, $\mathbb{S}^1 \times \mathbb{R}^{m-1}$ is the only possibility in this case.

(2) Continuing on, we could consider a 2-D shrinker times $\mathbb{R}^{m-2}$, contributing at most $m-2$, and so on.

Putting this together with the stratification theorem implies the above result. $\square$

6 Level set flow and Brakke flow

6.1 The avoidance principle for Brakke flows

Theorem 6.1. *Suppose M is the space-time support of an m-dimensional integral Brakke flow $(\mu_t)_{t\in I}$ in $U \subset \mathbb{R}^N$. Let $u : U \times I \to \mathbb{R}$ be a smooth function, so that at (x_0, t_0),*

$$\frac{\partial u}{\partial t} < tr_m \nabla^2 u\,,$$

where $\nabla^2 u$ is the spacial ambient Hessian, and tr_m is the sum of the smallest m eigenvalues. Then

$$u\big|_{M\cap\{t\le t_0\}}$$

cannot have a local maximum at (x_0, t_0).

Proof. Assume otherwise. We may assume that $M = M \cap \{t \le t_0\}$ and that $u|_M$ has a strict local maximum at (x_0, t_0). (Otherwise we could replace u by $u - (d(x, x_0))^4 - |t_0 - t|^2$).

Let $P(r) = B_r(x_0) \times (t_0 - r^2, t_0]$. Choose $r > 0$ small enough so that $-r^2$ is past the initial time of the flow, $u|_{\mathcal{M}\cap\overline{P}(r)}$ has a maximum at (x_0, t_0) and nowhere else and $\frac{\partial u}{\partial t} < \mathrm{tr}_m \nabla^2 u$ on $\overline{P}(r)$. By adding a constant we can furthermore assume that $u_{\mathcal{M}\cap(\bar{P}\setminus P)} < 0 < u(x_0, t_0)$. We let $u^+ := \max\{u, 0\}$ and plug $(u^+)^4$ into the definition of Brakke flow. Thus

$$
\begin{aligned}
0 &\le \int_{B_r} (u^+)^4 \, d\mu_{t_0} = \int_{B_r} (u^+)^4 \, d\mu_{t_0} - \int_{B_r} (u^+)^4 \, d\mu_{t_0 - r^2} \\
&\le \int_{t_0 - r^2}^{t_0} \int \left(\frac{\partial}{\partial t} (u^+)^4 + \langle \mathbf{H}, \nabla (u^+)^4 \rangle - |\mathbf{H}|^2 (u^+)^4 \right) d\mu_t \, dt \\
&\le \int_{t_0 - r^2}^{t_0} \int \left(\frac{\partial}{\partial t} (u^+)^4 - \mathrm{div}_{\mathcal{M}(t)} \big(\nabla (u^+)^4 \big) \right) d\mu_t \, dt \\
&= \int_{t_0 - r^2}^{t_0} \int 4 \bigg((u^+)^3 \frac{\partial}{\partial t} u^+ - 3 (u^+)^2 |\nabla^{\mathcal{M}(t)} u^+|^2 - (u^+)^3 \mathrm{div}_{\mathcal{M}(t)} \big(\nabla (u^+) \big) \bigg) d\mu_t \, dt \\
&\le \int_{t_0 - r^2}^{t_0} \int 4 (u^+)^3 \left(\frac{\partial}{\partial t} u^+ - \mathrm{tr}_m \nabla^2 u^+ \right) d\mu_t \, dt < 0 \, ,
\end{aligned}
$$

which is a contradiction. □

As a consequence of this, we obtain

Theorem 6.2 (Weak barrier principle)**.** *Let M be the space-time support of an m-dimensional integral Brakke flow in $U \subset \mathbb{R}^N$. Suppose that $t \mapsto N(t)$ is a 1-parameter family of domains in U so that $t \mapsto \partial N(t)$ is a smooth 1-parameter family of hypersurfaces. Assume that $M(t) := \{x : (x,t) \in M\} \subset N(t)$.*

If $p \in M(\tau) \cap \partial N(\tau)$, then $v(p,\tau) \ge h_m(p,\tau)$, where $v(p,\tau)$ is the speed of $\partial N(\tau)$ at p in the inward direction ν and h_m is the sum of the m smallest principal curvature of ∂N.

Proof. Let $f : U \to \mathbb{R}$ be defined by

$$
f(x,t) = \begin{cases} -\mathrm{dist}(x, \partial N(t)) : x \in N(t) \\ \mathrm{dist}(x, \partial N(t)) : x \notin N(t) \end{cases}
$$

and let $e_1, \ldots, e_{N-1}$ denote the principal curvature directions of $\partial N(t)$ at p. Then

$e_1, \ldots, e_{N-1}, \nu$ is an orthonomal basis at p. We compute

$$D^2 f(p) = \begin{pmatrix} \kappa_1 & & & \\ & \ddots & & \\ & & \kappa_{N-1} & \\ & & & 0 \end{pmatrix}.$$

Set $u = e^{\alpha f}$. Then $Du = \alpha e^{\alpha f} Df$, so

$$D^2 u = \alpha^2 e^{\alpha f} Df^T Df + \alpha e^{\alpha f} D^2 f .$$

From this, we readily see that the eigenvalues of $D^2 u$ at p (note that $u(p) = 1$) are $\alpha\kappa_1, \ldots \alpha\kappa_{N-1}, \alpha^2$. For α sufficiently large, we see that

$$\mathrm{tr}_m D^2 u|_p = \alpha h_m .$$

On the other hand,

$$\frac{\partial u}{\partial t} = \alpha e^{\alpha f} \frac{\partial f}{\partial t} = \alpha v(p,t) .$$

By assumption $f|_M$ has a maximum at (p,t), so the conclusion follows from the maximum principle proven above. □

Theorem 6.3 (Barrier principle for hypersurfaces)**.** *Let M be the space-time support of an n-dimensional integral Brakke flow in $\mathbb{R}^{n+1}$ and let $M(t) := \{x : (x,t) \in M\}$ denote the t-time slice of M. Suppose that $t \mapsto N(t)$ is a 1-parameter family of closed domains so that $t \mapsto \partial N(t)$ is a smooth 1-parameter family of hypersurfaces. Assume that $\partial N(t)$ is compact and connected and $v_{\partial N,in} \leq H_{\partial N,in}$ everywhere. Suppose that $M(0) \subset N(0)$ and that $\partial N(0) \setminus M(0)$ is nonempty. Then, $M(t)$ is contained in the interior of $N(t)$ for $t > 0$.*

First we prove

Lemma 6.4. *Assumptions as in the barrier principle. If $M(0) \subset N(0)$, then $M(t) \subset N(t)$.*

Proof. Let $\tilde{N}_\varepsilon(t)$ be the region with $\partial N_\varepsilon(0) = \partial N(0)$ and which flows with speed $H - \varepsilon$. If ε is sufficiently small, this flow will be smooth on an interval comparable to that of the definition of $N(t)$. We can apply the weak maximum principle and then let $\varepsilon \to 0$. □

Proof of Theorem 6.3. Since we assume that $\partial N(0)$ has some points which are disjoint from $M(0)$, we may push $\partial N(0)$ slightly in at these points, to find a new set $\hat{N}(0)$ with $\hat{N}(0)$ smooth, and $M(0) \subset \hat{N}(0) \subsetneq N(0)$. Flow $\partial \hat{N}(0)$ by mean curvature flow; it will remain smooth for at least a short time. The classical maximum principle shows that $\partial N(t)$ and $\partial \hat{N}(t)$ immediately become disjoint. Applying the above lemma to $\hat{N}(t)$ yields the desired result. □

Remark 6.5: This only works in codimension one. Nevertheless one can see from the weak barrier principle that in higher codimension spheres with Radius $R(t) = \sqrt{R_0^2 - 2mt}$ act as barriers from the inside and from the outside.

We will now extend this to an avoidance principle for Brakke flows. We first need the following auxiliary result of Ilmanen.

Lemma 6.6 ($C^{1,1}$-interposition Lemma, [21, Lemma 4E])**.** *Given disjoint closed sets $X, Y \subset \mathbb{R}^n$, X compact, there exists a compact $C^{1,1}$ hypersurface Q and a bounded open set U such that*

(i) $X \subset U$, $Q = \partial U$, $Y \subseteq \mathbb{R}^n \setminus \overline{U}$,

(ii) $\mathrm{dist}(X, Q) + \mathrm{dist}(Q, Y) = \mathrm{dist}(X, Y)$.

This immediately implies

Theorem 6.7 (Avoidance for codimension one Brakke flows)**.** *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two n-dimensional integral Brakke flows on $\mathbb{R}^{n+1}$ defined for $t \geq 0$ and such that* $\mathrm{spt}(\mathcal{M}_1(0))$ *is compact and* $\mathrm{spt}(\mathcal{M}_1(0)) \cap \mathrm{spt}(\mathcal{M}_2(0)) = \emptyset$. *Then*

$$[0, \infty) \ni t \mapsto \mathrm{dist}(\mathrm{spt}(\mathcal{M}_1(t)), \mathrm{spt}(\mathcal{M}_2(t)))$$

is strictly increasing.

Proof. Using big spheres as barriers we see that $\mathrm{spt}(\mathcal{M}_1(t))$ is compact for all $t > 0$. By Lemma 6.6 we can choose a closed domain N such that ∂N is a $C^{1,1}$-hypersurface such that

(i) $\mathrm{spt}(\mathcal{M}_1(0)) \subset \mathrm{int}(N)$, $\mathrm{spt}(\mathcal{M}_2(0)) \subseteq \mathbb{R}^{n+1} \setminus N$,

(ii) $\mathrm{dist}(\mathrm{spt}(\mathcal{M}_1(0)), \partial N) + \mathrm{dist}(\partial N, \mathrm{spt}(\mathcal{M}_2(0))) = (\mathrm{spt}(\mathcal{M}_1(0)), \mathrm{spt}(\mathcal{M}_2(0)))$.

Let $N(t)$ be the region with $\partial N(0) = \partial N$, flowing by mean curvature. Since ∂N is $C^{1,1}$ this will exist for a short time. We can then use Theorem 6.3 to see that

$$\begin{aligned}\operatorname{dist}(\operatorname{spt}(\mathcal{M}_1(0)), \operatorname{spt}(\mathcal{M}_2(0))) &= \operatorname{dist}(\operatorname{spt}(\mathcal{M}_1(0)), \partial N) + \operatorname{dist}(\partial N, \operatorname{spt}(\mathcal{M}_2(0))) \\ &< \operatorname{dist}(\operatorname{spt}(\mathcal{M}_1(t)), \partial N(t)) + \operatorname{dist}(\partial N(t), \operatorname{spt}(\mathcal{M}_2(t))) \\ &\leq \operatorname{dist}(\operatorname{spt}(\mathcal{M}_1(t)), \operatorname{spt}(\mathcal{M}_2(t)))\end{aligned}$$

for $t > 0$. □

6.2 Level set flow

We will give a brief introduction to level set flow and discuss some connections with Brakke flow, compare [21, 22].

Definition 6.8. *A family $\{\Delta_t\}_{t\geq 0}$ of closed sets in $\mathbb{R}^{n+1}$ is a* set-theoretic subsolution *of mean curvature flow, provided that for any family $\{M_t\}_{t\in[t_0,t_1]}$ of smooth, closed hypersurfaces moving by mean curavture,*

$$\Delta_{t_0} \cap M_{t_0} = \emptyset \qquad \textit{implies} \qquad \Delta_t \cap M_t = \emptyset \quad \textit{for all } t \in [t_0, t_1]$$

or equivalently,

$$\textit{dist}(\Delta_t, M_t) \geq \textit{dist}(\Delta_{t_0}, M_{t_0}) \quad \textit{for } t \in [t_0, t_1]\,.$$

Note that the equivalence follows from the translation invariance of mean curvature flow.

By using the $C^{1,1}$-interposition Lemma as in the proof of the avoidance principle for Brakke flows we obtain

Lemma 6.9 (Avoidance Lemma for set-theoretic subsolutions)**.** *Let $\{\Delta_t\}_{t\geq 0}$, $\{\Gamma_t\}_{t\geq 0}$ be set-theoretic subsolutions of mean curvature flow in $\mathbb{R}^{n+1}$. Assume*

$$\Delta_0 \cup \Gamma_0 = \emptyset, \quad \Gamma_0 \textit{ compact.}$$

Then $t \mapsto \textit{dist}(\Delta_t, \Gamma_t)$ is non-decreasing.

Note that the union of set-theoretic subsolutions is trivially again a subsolution. Thus we can define

Definition 6.10 (Level-set flow)**.** *Let $\Delta \subset \mathbb{R}^{n+1}$ be closed. The level-set flow $\{F_t(\Delta)\}_{t\geq 0}$ of Δ is the* maximal set-theoretic subsolution *starting such that $\Delta_0 := F_0(\Delta) = \Delta$.*

Proposition 6.11 (Basic properties)**.** *The level-set flow is well-defined and unique, and has the following basic properties*

- *semigroup property: $F_0(\Delta) = \Delta, F_{t+t'}(\Delta) = F_t(F_{t'}(\Delta))$,*
- *commutes with translations: $F_t(\Delta + x) = F_t(\Delta) + x$,*
- *containment: if $\Delta \subseteq \Delta'$, then $F_t(\Delta) \subseteq F_t(\Delta')$.*

Proof. Observe first that by translation-invariance of smooth solutions, a family of closed sets $\{\Delta_t\}$ is a subsolution if and only if

$$d(\Delta_t, M_t) \geq d(\Delta_{t_0}, M_{t_0}) \quad \forall\, t \in [t_0, t_1]\,,$$

whenever $\{M_t\}_{t\in[t_0,t_1]}$ is a smooth closed mean curvature flow. Now, considering the closure of the union of all subsolutions, namely

$$F_{t'}(\Delta) = \overline{\bigcup \{\Delta_{t'} \mid \{\Delta_t\}_{t\geq 0} \text{ is a subsolution}\}}\,,$$

we see that the level-set flow exists and is unique. The basic properties follow from existence and uniqueness. □

Relation to level-set-flow as defined by Evans-Spruck and Chen-Giga-Goto, see [21]. In [14, 4], the following equation appears together with a (viscosity) definition of its weak solutions

$$(\star) \qquad \begin{cases} \partial_t u = (\delta_{ij} - \nu_i \nu_j)\nabla^2_{ij} u & \text{on } \mathbb{R}^{n+1} \times [0, \infty) \\ u(\cdot, 0) = f(\cdot) & \text{on } \mathbb{R}^{n+1} \times \{0\}, \end{cases}$$

where $\nu = Du/|Du|$. When u is smooth and $Du \neq 0$, equation($\star$) says that the level-sets of u are simultaneously moving by mean curvature. If f is continuous and all but at most one of the level-sets of f are compact, then there exists a unique u weakly solving ($\star$) (see [20, §7]). The family of level-sets $\Gamma^a_t := \{x : u(x,t) = a\}, t \geq 0$, is unique and is called the *level-set flow* (by mean curvature) of $f^{-1}(a)$. It follows trivially from the definition of weak solutions of ($\star$) (which involves tangency of u with smooth test functions, see [14]) that a level-set flow in $\mathbb{R}^{n+1}$ is a set-theoretic subsolution of mean curvature flow.

In fact, any set-theoretic subsolution that is contained within a level-set flow at $t = 0$ must remain contained within it, because otherwise it would run into some of the other level-sets if u, violating the avoidance of set-theoretic subsolutions. (Note that always one of the two sets involved in the collision is compact by the hypothesis on f). This shows the equivalence of the two definitions.

Relation to Brakke flow. From the barrier principle for hypersurfaces, Theorem 6.3 we have

Lemma 6.12. *Let M be the space-time support of an n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$. Then the family of sets $M(t) = \{x : (x,t) \in M\}$ are a set-theoretic subsolution of mean curvature flow.*

Corollary 6.13. *Let $\mathcal{M}$ be n-dimensional integral Brakke flow on $\mathbb{R}^{n+1}$ for $t \in [0,\infty)$ and let $\{\Gamma_t\}_{t\geq 0}$ be a level set flow. Then*

$$\operatorname{spt}\mu_0 \subset \Gamma_0 \quad \textit{implies} \quad \operatorname{spt}\mu_t \subset \Gamma_t$$

for all $t \geq 0$.

This implies that one can use the level-set flow to characterise possible non-uniqueness of possible Brakke flows starting at M_0. The notion used for this is called *non-fattening.*

Definition 6.14. *A $\{\Gamma_t\}_{t\geq 0}$ level-set flow is called* non-fattening*, provided*

$$\mathcal{H}^{n+2}\Big(\bigcup_{t\geq 0}\Gamma_t \times \{t\}\Big) = 0\,.$$

Note that this implies that for the level-set flow $\{\Gamma^a_t\}_{t\geq 0}$ as defined by Evans-Spruck or Chen-Giga-Goto at most countably levels $\{u(x,t) = a\}$ could be fattening. This shows that non-fattening is a generic condition. We have the following equivalence:

Lemma 6.15 ([22, §11.4], [15, 4.2])**.** *If $\mathcal{H}^n(\Gamma_0) < \infty$, then the level set flow $\{\Gamma_t\}_{t\geq 0}$ is non-fattening if and only if $\mathcal{H}^n(\Gamma_t) < \infty$ for all $t \geq 0$.*

Remark 6.16: It follows by work of Hershkovits-White [18] and the resolution of the mean-convex neighborhood conjecture by Choi-Haslhofer-Hershkovits [6] and Choi-

Haslhofer-Hershkovits-White [7] that if a unit-regular, cyclic mod 2, n-dimensional integral Brakke flow $\mathcal{M}$ in $\mathbb{R}^{n+1}$, starting at a compact, smooth, embedded hypersurface M_0 has only multiplicity one spherical and neck-pinch (i.e. of type $\mathbb{S}^{n-1} \times \mathbb{R}$) singularities, then the level-set flow of M_0 is non-fattening. It then follows from [5, Corollary F.5] that the unit-regular, cyclic mod 2, n-dimensional integral Brakke flow starting at M_0 is unique.

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