THE WILMORE CONJECTURE AND YAU'S CONJECTURE

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Abstract. Informal notes for the learning seminar at the GeoTop centre in Copenhagen, spring '25 on centrally important recent (past 10 years or so) advances in geometric analysis by minmax. No great claims of originality, of course, and there are already many great surveys on these results available but hopefully they serve as an enjoyable introduction and interest the reader in delving deeper into the literature. A familiarity with geometric analysis and minimal surfaces in particular is assumed.

Contents

1. A (very) brief reminder of the minmax method

Earlier in the learning seminar we already saw, from Harish Upadhyaya, a detailed account of minmax ala Simon–Smith, following Colding–DeLellis, but I figure if I'm going to go through the trouble of writing up something I might as well start from the beginning. In a introductory PDE course (maybe the second semester) at some point one finds out about the **direct method**, which is a way to find solutions to variational problems which correspond to critical points of an energy functional. Denoting our space of objects X (Sobolev space for instance) and an energy functional $I : X \to \mathbb{R}$ to find a critical point what one does is to consider a limit of $x_i \in X$ where $I(x_i)$ converge to the infimum of the energy I on X. To make this work, we need the following ingredients:

(1) That the infimum of I is bounded below by some constant.

(2) Appropriate compactness theorems on X to by able to take a converging subsequence of the x_i – this can be a sticky point because typically we will consider noncompact (even typically infinite dimensional) spaces of objects. For instance if $X = H^2(U)$ one employs Relich–Kodrachov compactness theorem. In order to satisfy the conditions of this theorem we in turn need boundedness of x_i in Sobolev norm, which means that we should be able to estimate $||x_i||$ in terms of $I(x_i)$ (but now we are getting a bit particular).

(3) Supposing we can take a converging subsequence of the x_i to get a limit x, we want to know that $I(x)$ achieves the infimum of the energy. In general I might not be continuous so this is again not always obvious. At the very least though we need that the energy is lower semicontinuous, which means that it can only drop in the limit, and this is guaranteed in many (most?) important cases.

If we have all the conditions above met we can find a solution, although as you recall (from the utterance Sobolev spaces) for X to be large enough to be complete and have satisfy good compactness results the solution we find may only be a weak solution. For instance if we are trying to use the direct method to solve a PDE like $-\Delta u = f$ we would use the energy $I[u] = \int_U |\nabla u|^2 - uf$ and a solution from above then would be a critical point of $I[u]$, which is great, but we don't know apriori know it even has two derivatives so could solve $-\Delta u = f$ "classically." Hence we need to also establish a **regularity theory** for critical points found by this method, which is to say that critical points $u \in H^2$ of I actually lay in the smaller space C^2 (how smooth actually depends on f). This is usually the hard part for these sorts of schemes.

In these informal notes, our goal is to discuss finding critical points (and consequences thereof) of the area functional, which correspond to minimal surfaces; here X is roughly speaking the space of varifolds and currents and the regularity theory can be quite involved but it often works. One can certainly find minimal surfaces via the direct method: a classical result that in any $n-1$ homology class one can find a minimal representative. But generally speaking if you apply the direct method you might get an empty set – an uninteresting critical point. And of course this is for a very good reason; minima of the area functional will be stable minimal surfaces, but as you know from the second variation formula in many cases there are none (when $\text{Ric} > 0$).

We might still want to find/use minimal surfaces though, and minmax provides a way to find nonstable critical points/saddle points for the area functional. The idea in pictures, although I'm too lazy to draw a picture here, is first to visualize a saddle and consider a path going P from one side (where one stirrup is) to the other side. Denoting by M_P to be the maximum height of P along the saddle, If you think about it for a minute the (height of) the saddle point of the saddle is the smallest value of M_P for all such paths P. In other words:

$$\text{height of saddle point} = \inf_{\text{paths } P} \sup_{p \in P} \text{height}(p)$$

Hence the term minmax. For minimal surfaces our "saddle" is a manifold M, and paths correspond to "sweepouts" of M by surfaces, a sweepout being some sort of generalized foliation of M by surfaces. This is the Almgren–Pitts minmax theory. In the rest of this section we will restrict ourselves to the Simon–Smith variant of Almgren–Pitts minmax theory, because that is what we learned about earlier on from Harish's talks; this you could say is a more smooth version of minimax and is for finding minimal surfaces in 3–manifolds. A nice property of it is that the topology of the minmax limit is more easily controlled in this scheme. Sometimes throughout these notes (i.e. after the Wilmore conjecture) we will need more properly Almgren–Pitts but for the sake of brevity will treat it a a black box as much as possible – the general method is the same. Moving on, we may consider some initial god–given sweepout S (for instance, the one on S^2 by round circles including two points at north and south poles), and then by "all paths" we mean images of S under all isotopies – this is called the saturation Λ of a sweepout. In this setting the "height" is called the width of the saturation, and we have as above:

$$\text{width}(\Lambda) = \inf_{\text{sweepouts } S' \in \Lambda} \sup_{\text{slice } \Sigma \in S'} \text{Area}(\Sigma)$$

Note that because the original sweepout is included in the saturation bounds on areas for it imply bounds on the width above. The big statement, of course, is that often there actually is a surface actually realizing width(Λ), that it is nontrivial (i.e. not the empty set), that it is minimal, and that it is smooth (modulo some dimensional restrictions related to regularity theory for minimal surfaces, let's just suppose here that $n = 3$). To show that the surface, if it exists, is nontrivial amounts to showing the width is nonzero, and this is typically achieved using the isoperimetric inequality or some topological condition. Working in the space of varifolds, one can apply a compactness argument like in the direct method to see that there is at least a varifold V achieving the width. To see that it is a stationary varifold, a sort of discretized mean curvature flow can be used to show that if $H \neq 0$ then there is a surface $\Sigma' \in \Lambda$ with area$(\Sigma') <$ width(Λ) giving a contradiction. The regularity theory, naturally, is the most difficult part and uses that Λ contains all isotopies of sweepouts comprising it. One can show that away from finitely many points V is "almost" minimizing, and so by using a smooth competitor from Meeks–Simon–Yau (plateau problem in isotopy class) one can see that V is smooth away from finitely many points. A removable singularity theorem then finishes the job.

One can also show that the genus of the limit can only decrease relative to the surfaces in the sweepout; this might seem obvious but actually is not because, getting into technicalities, the varifold topology is quite weak (see the introduction of DeLellis–Pelladini for a example). An upshot is that and important early triumph of minmax is the following:

Theorem 1.1 (Simon-Smith). *Every 3–sphere (S^3, g) contains an embedded minimal 2–sphere Σ.*

Of course there may be many more minimal spheres, like in the case of the round sphere. A natural question, then, is how many should we usually expect? One perspective, which isn't always quite accurate but good to keep in mind, is to imagine the area functional as a Morse function on the space of surfaces (sloppily speaking, although this will be made more precise below), so studying the topology of the space of surfaces should lend some insight. This brings us to the following detour, which gets us thinking a bit in the direction of Yau's conjecture:

1.1. **How many critical points? The Lusternik–Schnirelmann category.** The LS category gives a way to lower bound the number of critical points of *any* smooth function on a manifold though. Its defined in terms of open covers: the Lusternik–Schnirelmann category of a manifold M, denoted by $LS(M)$, is the smallest number need to cover M by contractible open subsets.

By a contractible open set U we mean specifically that there is a map f so that $f : M \to M$ so that f is homotopic to the identity and maps U to a point in M. So for instance if $LS(M) = 1$ then M is contractible, and if its 2 its a sphere. We relate it to the topology in terms of deRham cohomology for the sake of familiarity:

Theorem 1.2. *$CL(M) + 1 \leq LS(M)$, where CL is the cup length.*

Proof: By cup length here we mean the maximal number of (positive degree, closed) forms ω_i such that their wedge is nonexact. Denoting $LS(M)$ by k, we consider k forms ω_1 through ω_k and wish to show their product $\omega_1 \wedge \cdots \wedge \omega_k$ is exact – to calculate the idea is to work in the f_i above somehow. Now, because each of the U_i are contractible and are of positive degree, we have $f_i^*\omega_i = 0$ restricted to U_i (by Poincare lemma ω_i is exact on this set, and since pushforward commutes with d $f_i^*\omega_i = 0$). Because the U_i cover M then we have $\bigwedge f_i^*\omega_i = 0$. One the other hand since the f_i are homotopic to the identity there are forms θ_i so that $\omega_i = f_i^*\omega_i + d\theta_i$ (i.e. cohomology classes are homotopy invariant). Because $\bigwedge f_i^*\omega_i = 0$, we then have that $\omega_1 \wedge \cdots \wedge \omega_k$ is the sum of products $\beta_1 \wedge \cdots \wedge \beta_k$ where at least one of the $\beta_i = d\theta_i$. Because the ω_i (and hence pushforwards of them) are all closed, this gives that each of these products is exact by the product rule, giving the claim. □

The next fact relates the category to the number of fixed points of a function:

Theorem 1.3. *Suppose M is a compact (closed) smooth manifold and $f : M \to \mathbb{R}$ is a smooth function. Then $LS(M)$ is bounded above by the number of critical points of f.*

Proof: Without loss of generality, the number of critical points is finite. Let $p_1, \ldots p_k$ be the critical points of f, to these critical points we will give k contractible sets. Consider a small (ambiently contracitble) ball B_i about each p_i, and consider (w.r.t. some background metric) the flow of them by the gradient of f; write the image of B_i under it up to time m by $\phi^m(B_i)$. Then $M = \cup_i \cup_{m=0}^{\infty} \phi^m(B_i)$. By the compactness of M, there exists some $T >> 0$ so that $M = U_1 \cup \cdots \cup U_k$, where $U_k = \cup_{0 \leq t \leq T} \phi^T(B_i)$ (each of these for fixed T are open since the flow is ran for just a finite time). Each of them are contractible since B_i is, giving the upper bound on the category. □

So, for any smooth function we have a lower bound on its number of critical points in terms of the cup length of the manifold it resides on. A nice way to apply it is to predict the number of minimal surfaces a manifold of a topological type should have. I write predict here, because in the technical framework some critical points might be counted with multiplicity and so aren't interesting from a geometric perspective and this is a central issue in using minmax to find many minimal surfaces. For instance, by the resolution of the Smale conjecture it turns out that the space of embedded 2-spheres $\mathcal{G}$ in S^3, allowing for degenerations, has $\mathcal{G}/\partial\mathcal{G} \sim \mathbb{R}P^4$ (we will see a related calculation below when we more formally discuss Yau's conjecture). Since the cohomology ring of this is $\mathbb{R}[\alpha]/(\alpha^5)$ the cup length is 4. The zero area critical point is noninteresting, so we get the following conjecture:

Conjecture 1.1. *In any (S^3, g) there should be at least 4 minimal 2–spheres.*

The first result along this line after Simon–Smith, to my knowledge, was by Haslhofer and Ketover in 2019 where they showed that any bumpy sphere (i.e. no Jacobi fields) has at least two minimally embedded spheres, and much more recently Z. Wang and Zhou confirmed the conjecture in a bumpy metric.

Now note that to produce different minimal surfaces one has to start with different sweepouts, which depending on the tools at hand and setting might require some real creativity to get something "interesting." For instance in her talks Priya Kevari discussed how Haslhofer and Ketover use the mean curvature flow to produce useful sweepouts along with Ketover's catenoid estimate. The idea, in a nutshell, is that taking the unstable minimal sphere Σ_{SS} in (S^3, g) produced from Simon–Smith (morally speaking, since we are finding saddle points, the minmax limits usually should be unstable) one can perturb Σ_{SS} on either side (i.e. flipping normals) by the first eigenfunction of its Jacobi operator to get two nearby mean convex spheres. The (weak) mean curvature flow of these will emanate away from Σ_{SS}, and, after some reductions i.e. dealing with the case there are some other stable minimal spheres, using the flow of these we can produce a sweepout Φ of (S^3, g), which will be by connected slices by joining them with a thin neck of controlled area.

In particular one can arrange that the areas of the leaves are less than $2\text{area}(\Sigma_{SS})$; this tells us that the minmax limit cannot be $2\Sigma_{SS}$. The sweepout Φ, where essentially one varies distance of the slices (just of the flow of the perturbations) from Σ_{SS} and neck width, is a two parameter family so if minmax applied to this sweepout gives Σ_{SS} with multiplicity one then it turns out, as discussed in lemma 4.7 that there are in fact infinitely many embedded spheres completing the sketch.

Now as you gather from the discussion above a major issue is that one may use all sorts of different sweepouts and apply the minmax machine, but the minimal surface found might be the same one but with higher multiplicity – not geometrically distinct in a sense. We will return to this point multiple times below.

2. An interlude into geometric inequalities via minmax: the Wilmore conjecture

Many of the more recent results we discuss in these notes are due to or connected in some way to the fantastic contributions of Marques and Neves to the theory and applications of minmax, and to be chronologically accurate to their progress we'll start with the Wilmore conjecture. It is also interesting to see how the minmax method can be used to do something more than "just" finding lots of minimal surfaces – these things are good for something sometimes!

The Wilmore energy $W(\Sigma)$ of a surface Σ in $\mathbb{R}^3$ is given by $\int_\Sigma H^2 dx$, and represents in some sense the totality of how much a surface bends (here using H averaged). Of course there are a number of such quantities one may consider, but the Wilmore energy does appear in some real life models and in fact there is a "proof" of the Wilmore conjecture from biology. Wilmore originally showed the Wilmore inequality, which is that:

Theorem 2.1. *For every compact surface Σ in $\mathbb{R}^3$, $W(\Sigma) \geq 4\pi$ with equality if Σ is a round sphere.*

Proof: This of course isn't to be confused with the Wilmore conjecture, but this is still interesting and there are a number of proofs of this. Considering the plane note that compactness in necessary; now to see this note that the Gauss map $N : \Sigma \to S^2$ is onto for all compact surfaces, and that in fact restricting to the set P for which $K \geq 0$ we have by the moving plane argument $N\ |_P \colon \Sigma \to S^2$ is onto. Since $K = \det(dN)$, the multivariate change of variables formula says then that $\int_P K \geq \text{area}(S^2) = 4\pi$ (on this set N is local diffeo). Because $H^2 \geq K$, we get the claim. $H^2 = K$ precisely at umbilical points gives the rigidity statement. □

With that out of the way a natural next question is to examine if anything more can be said under some topological restrictions on Σ. Wilmore studied families of tori given by surfaces of revolution and, perhaps hastily, made his conjecture:

Conjecture 2.1 (Wilmore)**.** *Every compact surface Σ of genus one in $\mathbb{R}^3$ has $W(\Sigma) \geq 2\pi^2$.*

Also, considering how symmetric the Wilmore energy is maybe considering just a specific set of examples isn't so bad. For instance the Wilmore energy is invariant under conformal transformations so its is equvalent to a statement for surfaces in $S^3(1)$ via stereographic projection. There the Wilmore conjecture is:

Conjecture 2.2 (Wilmore)**.** *Every compact surface Σ of genus one in S^3 has $\mathcal{W}(\Sigma) \geq 2\pi^2$, where $\mathcal{W}(\Sigma) = \int_\Sigma 1 + H^2 dx$.*

In the class of rotationally symmetric tori Wilmore considered there is, as one might expect, a torus for which equality is obtained. Under stereographic projection this is a minimally embedded torus C with $W(C) = 2\pi^2$; $C = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3(1)$ is called the Clifford torus and will play an important role below.

Now, as one can read about in depth (and they are certainly worth the read!) in any of the easily found survey articles of Marques and Neves a great number of famous mathematicians have made progress on the problem, in various special circumstances (different symmetry conditions and the like). One useful fact, due to Li and Yau, is that for the purposes of the conjecture it suffices to consider only embedded surfaces. This is also a good time to point out that the Wilmore energy is an interesting functional of study in its own right (as minimal surfaces are), and there are many interesting questions one could consider about it. Its still an active field of study! Critical points of W are called Wilmore surfaces, and by the first variation formula satisfy

$$\Delta H + \frac{(k_1 - k_2)^2}{2} H = 0 \tag{2.1}$$

In particular all minimal surfaces are Wilmore surfaces – for WIlmore's evidence this had better be the case or else there is no way he found a minima! There are Wilmore surfaces which aren't minimal though. It does seem to be the case that Wilmore surfaces (and the Wilmore flow) are less studied than the minimal surfaces and the MCF and perhaps this can be attributed to the fact that the equation above is, considering that $H = \Delta X$, a fourth order PDE. Much less is known about these sorts of PDE largely (I believe) because the maximum principle often cannot be applied.

Luckily though to show the Wilmore conjecture via minmax it suffices to stay in the realm of minimal surfaces. The central fact about the Clifford torus, besides that $\mathcal{W}(C) = 2\pi^2$, is the following theorem of Urbano (and apparently in unpublished work Fischer-Colbrie):

Theorem 2.2. *Let M be a compact orientable non-totally geodesic minimal surface in $S^3(1)$. Then $ind(M) \geq 5$, with equality exactly when M is the Clifford torus.*

Proof: [particularly sketchy sketch] The index is at least 4 if M is not totally geodesic, from the isometries of $S^3 \subset \mathbb{R}^4$ namely translations by considering the functions $f_a = \langle N, a\rangle$, $a \in \mathbb{R}^4$. These will be eigenfunctions of eigenvalue -2 so that there must be an additional simple eigenvalue, giving that the index is always at least 5. If the index is actually equal to five, then using the conformal balancing trick one can show that $2A \geq \int_M |A|^2$. Gauss formula gives $|A|^2 = 2 - 2K$ for minimal surfaces in the sphere, so that Gauss Bonnet implies M is a sphere or torus. The minimal spheres of S^3 are totally geodesic, so that M is a torus. One can also see that on M in fact $|A|^2 = 2$ implies that M is a Clifford torus, giving one direction. If M is a Clifford torus one can likewise see it has index 5 using that on it $L = \Delta + 4$. □

As an aside, indeed one can check that the totally geodesic minimal surfaces, being the great spheres, have index 1. Now, if one thinks about the toy picture a bit applying the minmax proceedure with a k–parameter sweepout should morally give a minimal surface with index less than or equal to k (in the saddle picture there are k "downward" directions from the saddle point) . With this in mind the scheme of Marques and Neves rests on two big hopes:

(1) Given a surface $\Sigma \subset S^3$, constuct a 5 parameter sweepout Γ so that for any $\gamma \in \Gamma$ $\mathcal{W}(\gamma) \leq \mathcal{W}(\Sigma)$.
(2) From paragraph above a minmax limit $\overline{\gamma}$ of Γ should give a great sphere or Clifford torus C. If C, then by continuity of Wilmore energy we have

$$2\pi^2 = \mathcal{W}(C) = \mathcal{W}(\overline{\gamma}) \leq \mathcal{W}(\Sigma) \tag{2.2}$$

If we can satisfy these two conditions, then what we will actually have shown the following:

Theorem 2.3 (Marques and Neves)**.** *Every embedded compact surface Σ in S^3 with positive genus satisfies*

$$2\pi^2 \leq \mathcal{W}(\Sigma) \tag{2.3}$$

There is also a rigidity statement, but let's not overcomplicate our lives here. Let me also point out that this result has useful implications in the singularity analysis of the mean curvature flow, it is not just a dead-end curiosity. Now we will discuss how to deal with (1) and (2) above and then to (hopefully!) reinforce the ideas apply them to show something in the simpler and perhaps more familiar setting of the mean curvature flow.

To deal with (1), a first natural source of inspiration is the symmetries of the Wilmore functional. Like we wrote before the Wilmore energy is invariant under conformal dilations of S^3; the conformal group of S^3 up to isometries is parameterized by the unit 4–ball B^4 where for each $v \in B^4$ we associate the conformal map:

$$F_v : S^3 \to S^3,\ F_v(x) = \frac{(1-|v|^2)}{|x-v|^2}(x-v) - v \tag{2.4}$$

Here F_0 is the identity map (since $|x| = 1$) and for $v \neq 0$ F_v is a conformal dilation with fixed points $v/|v|$ and $-v/|v|$. One can cook up these maps, and see that they are all of them, by taking conformal diffeomorphisms on R^3 and conjugating with stereographic projection.

This gives a 4 parameter family, but in order to obtain the Clifford torus from minmax again we should find a way to include another parameter. To get an extra parameter what Marques and Neves do is consider the signed distance to a surface: supposing a surface S bounds a domain Ω (which it will if its is embedded) we define S_t by:

$$\partial\{x \in S^3 \mid d(x,\Omega) \leq t\} \text{ if } 0 \leq t \leq \pi \tag{2.5}$$

and

$$\partial\{x \in S^3 \mid d(x, S^3 \setminus \Omega) \leq -t\} \text{ if } -\pi \leq t < 0 \tag{2.6}$$

At first this might seem a little strange but this is also quite natural, or at least I have a heuristic (just supposed to be provocative, get the people going, take it with a grain of salt): if we suppose that Σ is a minimal surface (the Clifford torus, for instance) then the area decreases when we perturb in the direction of the first eigenfunction, because every minimal surface of S^3 is unstable. The first eigenfunction is positive, so morally surfaces equidistant from Σ should have area less than Σ. Because at the end of the day by minmax we are really finding a minimal surface for which area and Wilmore energy are the same this suffices.

Denoting by $\Sigma_{(v,t)} = (F_v(\Sigma))_t$, indeed we have the following:

Theorem 2.4. *For any* $(v,t) \in B^4 \times (-\pi, \pi)$

$$\mathit{area}(\Sigma_{(v,t)}) \leq \mathcal{W}(\Sigma) \tag{2.7}$$

This 5 parameter set of surfaces is called the canonical family of Σ, and the theorem above says that it satisfies the properties we want; what remains is to bound the width below away from 4π. This is perhaps the most clever part of their proof and, considering starting with $\Sigma \simeq S^3$, should somehow involve the topology of Σ in a critical way. Because I think their mechanism to show this is so interesting and worth understanding, let's switch gears take a look at a similar but simpler sort of argument in the mean curvature flow setting:

2.1. **Ketover and Zhou's entropy bound.**

This will require a little bit of a setup, but should go quickly. First we quickly recall the definition of Colding and Minicozzi's entropy. The $F_{(x_0,r)}$ functionals are defined by:

$$F_{x_0,r}(\Sigma) = (4\pi r)^{-n/2} \int_\Sigma e^{\frac{-|x-x_0|^2}{4r}} d\mu \tag{2.8}$$

And the entropy of a surface Σ is defined as:

$$\lambda(\Sigma) = \sup_{x_0,r} F_{x_0,r}(\Sigma) \tag{2.9}$$

This has a number of interesting properties and is intensely studied in the mean curvature flow. Several important facts are:

(1) It is often finite, even for noncompact surfaces (vs the area)
(2) It is monotone nonincreasing under the flow, so that apriori bounds on the entropy of a surface can be used later on along the flow
(3) It is translation and dilation invariant, so that with (2) in mind entropy bounds on a surface imply bounds on shrinkers which may occur
(4) It has stable critical points, precisely the shrinking (generalized) round cylinders $S^k \times \mathbb{R}^{n-k}$, versus the Gaussian area (area in Gaussian metric, $g_{ij} = e^{-|x|^2/2n}\delta_{ij}$) – in $\mathbb{R}^3$ the shrinking sphere has index 4 essentially coming from translations and dilations so in some sense the definition of entropy ignores these.

Now an important early question about the entropy was whether it was bounded below; this is now known in great generality but Ketover and Zhou showed the then state of the art fact:

Theorem 2.5 (Ketover and Zhou)**.** *Suppose that $\Sigma \subset \mathbb{R}^3$ is an embedded sphere, then $\lambda(\Sigma) \geq \lambda(S^2)$.*

I can expound quite a bit here but for the sake of brevity perhaps wont – perhaps whats worth saying is that considering just running Σ under the flow and that round singularities are generic this should hold, so this inequality is a "test" of sorts. Of course the mean curvature flow has advanced quite a bit since then.

Getting to the point for these notes, the proof is by minmax in the Gaussian metric ala the Wilmore conjecture – there are some technical difficulties in carrying out minmax in this setting because the Gaussian metric is poorly behaved but we won't fret about it here. Here, the canonical family $\Sigma_{s,t} = s(\Sigma - t)$ we consider is built of translations and dilations of Σ so in fact $\lambda(\Sigma_{t,s}) = \lambda(\Sigma)$ for $(t, s) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$. Doing minmax with this family Γ, as in the Wilmore conjecture the hope is that any minmax surface $\overline{\gamma}$ associated to the family will satisfy $\lambda(\overline{\gamma}) \geq \lambda(S^2)$.

Of course the hope really is that γ is just the round sphere itself; from item (4) above this is at least plausible because we are considering a 4 parameter sweepout. In the Wilmore setting the mechanism to confine what sort of limit we saw was accomplished by Urbano's result; this setting the analogous ingredient is due to Brendle:

Theorem 2.6. *Suppose Σ is an embedded self shrinker of genus zero in $\mathbb{R}^3$. Then either it is a flat plane, round cylinder, or round sphere (of appropriate radius).*

Proof: [sketch] So, the idea is to show such a shrinker M must be mean convex. Supposing not, then there would be a point p where $H = 0$, so that $\langle x, \nu \rangle = 0$ there too by the shrinker equation. Denoting by a the unit normal taken at this point and Z the plane given by $\langle a, x \rangle = 0$, by nodal theory for elliptic PDE nearby p the intersection of Z and M must be a union of m curves intersecting transversely at p. To see this, setting $f = \langle a, x \rangle$ f is an eigenvalue of the drift laplacian on M, and the set $Z \cap M$ is the nodal set of this equation. Here m depends on the multiplicity of the zero of course, and by the choice of p and a one can see m must be at least 2. On the other hand, one can solve plateaus problem in the domains bounded by M with boundary Z and M, and one can see that the solution must be locally flat. The surface one finds is also embedded, so that m cant be 2 or more giving a contradiction. □

Below unless otherwise stated we sloppily write these as $\mathbb{R}^2$, S^2, and $S^1 \times \mathbb{R}$ respectively. It turns out that $\lambda(\mathbb{R}^2) = 1$ and $\lambda(S^2) \leq \lambda(S^1) = \lambda(S^1 \times \mathbb{R})$, so it suffices to rule out getting a plane as a minmax limit (and basically where we left off at the Wilmore conjecture) – on that note observe by zooming in about a point of Σ that $\lambda(\Sigma) > 1$. First, we discuss the behavior of the canonical family along the boundary of the parameter space and its asymptotic behavior

(1) For any $t \in \Sigma$, we have $\lim_{s \to \infty} \Sigma_{t,s} = T_t\Sigma(0)$, where $T_t\Sigma(0)$ is the tangent plane of $T_t\Sigma$ at the origin.
(2) For any $t \in \mathbb{R}^3 \setminus \Sigma$ we have $\lim_{s \to \infty} \Sigma_{t,s} = \emptyset$
(3) For any $t \in \mathbb{R}^3$, we have $\lim_{s \to 0} \Sigma_{t,s} = \emptyset$
(4) Finally, for any fixed s by the compactness of Σ we have $\lim_{|t| \to \infty} \Sigma_{t,s} = \emptyset$

These facts are all good because they say essentially that the entropy won't jump above $\lambda(\Sigma)$ when taking limits in the parameter space. For the sequel also note that in particular

$$\Sigma_{\cdot,\infty} : \Sigma \to S^2 \text{ is the Gauss map} \tag{2.10}$$

and the degree of the Gauss map is $1 - g$ which in particular, in our setting, is nonzero.

Considering items (1) and (2) above, an issue (which also appears in the Wilmore conjetcure) is that the sweepout is discontinuous along the top face $\mathbb{R}^3 \times \{\infty\}$ because it consists of either planes or the empty set. The first task of Ketover and Zhou then is to modify this family to make it continuous by "blowing up" along the boundary and modifying the family appropriately. Considering that $[0, \infty)$ is homeomorphic to $[0, 1)$ by the map $h(t) = \frac{2}{\pi}\tan^{-1}(t)$, we can consider the sweepout instead parameterized on $\mathbb{R}^3 \times [0, 1)$ by setting $\Sigma_{t,s} = \Sigma_{t,h^{-1}(s)}$ (overloading notation a bit). Then our goal is to extend the sweepout continuously to $\mathbb{R}^3 \times [0, 1]$. To do so, consider the tubular neighborhood

$$\Omega_\epsilon = \{x \in \mathbb{R}^3 \times [0, 1] \mid |x - (p, 1)| < \epsilon \text{ for some } p \in \Sigma\} \tag{2.11}$$

Now, for a point $(t_0, s_0) \in \partial\Omega_\epsilon$, for epsilon very small, the surface Σ_{t_0,s_0} will in a very large region be close to the tangent plane of $T_q\Sigma$ at the point q closest to t_0 and for ϵ small these planes would vary continuously along $\partial\Omega_\epsilon$; vaguely speaking (at least how I think about it) we could modify the sweepout in Ω_ϵ by defining $\Sigma_{t,s}$ to be these planes along appropriate curves connecting $\partial\Omega_\epsilon$ to $\Sigma \times 1$. Because planes have entropy equal to 1, the entropy of surfaces in this modified family will still be bounded above by $\lambda(\Sigma)$.

The stage is almost set; the topological ingredient in our argument is the following:

Lemma 2.7. *Let H be a closed handlebody in $\mathbb{R}^3$ with boundary a surface Σ of genus g. For $g \neq 1$, the reduced Gauss map $\tilde{G} : \Sigma \to \mathbb{R}P^2$ cannot extend to a continuous map defined on all of H.*

Proof: [sketch] Suppose such an extension was possible. The idea is that the degree, which is locally constant, of the Gauss map on a genus g surface has degree $1-g$ as we noted above so in this setting is nonzero. On the other hand because H deformation retracts onto a closed graph (1 dimensional) the degree would have to be equal to 0, giving a contradiction. □

Proof: [sketch of Theorem 2.5] Because Σ is embedded its separating and bounds a ball B (which is a handlebody, of course). Denoting by Λ_Σ the saturation associated to the caonical family, suppose that the width $W(\Lambda_\Sigma) = 1$. Then for each fixed $t \in B$ there is a sequence of one parameter sweepouts $\Phi_i(t, s) \subset \Lambda_\Sigma$ with maximal Gaussian areas approaching 1 (from above), which we can suppose are tightened. On the other hand by the isoperimetric inequality the Gaussian area is bounded below from 1. Hence for each i there is a value of s for which $\Sigma_{t,s}$ is very close to a signed plane. The normal to this plane gives a map to S^2, which on Σ corresponds with the Gauss map. Everything varies continuously so that we have a map which can't exist by the lemma above, giving a contradiction. □

3. What Yau's conjecture is and the topological rationale

After the Wilmore conjecture, Marques and Neves (and associates) attacked the Yau conjecture, originally stated as:

Conjecture 3.1. *Every compact 3-manifold admits an infinite number of smooth, closed, immersed minimal surfaces.*

A result along these lines was shown for geodesics on surfaces by Franks and Bangert, using ideas from dynamical systems. These methods don't work in higher dimensions though, which (if I recall from a talk by Marques) is a reason why Yau's conjecture is interesting. Its also interesting because there are lots of different ways (in various settings) to produce minimal surfaces, and Yau's conjecture is a yardstick of sorts to see how good they are. A non–exhaustive list of ways:

- Direct method – good for finding stable critical points.
- The mean curvature flow – again really only good for finding stable minimal surfaces and there are regularity issues with the flow.
- Weirstrass representation – classical complex analysis stuff, works in $\mathbb{R}^3$, throwing this in here just because.
- The gluing method – desingularize or double minimal surfaces using inverse function theorem. The relevant operator usually has kernel though and the spaces of functions involved are difficult to manage and works best in $n = 3$ but can produce unstable critical points.
- Minmax – what Marques–Neves and friends use.
- Allen–Cahn, nonlocal minimal surfaces, etc. We might refer to these has relaxation methods which have the commonality that one perturbs the area functional in some way to get into a situation where one has more powerful analysis techniques. These until recently were a bit less mainstream in geometric analysis (at least in geometric analysis as far as I know) compared to gluing and minmax but are very powerful. We'll find out more about these in the masterclass coming up.

So as you can gather the only two methods above that really only have a shot are gluing, minmax and relaxation methods. I'm pretty sure I've read in surveys that the gluing method does indeed work but I'm not sure on the status of this in the literature (I don't think there are reasons it shouldn't work it might just not be written down really yet). Allen–Cahn and nonlocal minimal surfaces work well but I don't want to say more here. Minmax has been enormously successful. The penultimate result was shown by Antoine Song, a student of Marques and Neves in a 2018 tour de force:

Theorem 3.1 (Song)**.** *In any closed Riemannian manifold of dimension $3 \leq n \leq 7$ there exist infinitely many smoothly embedded closed minimal hypersurfaces.*

The result above was shown already in the generic case, by Irie, Marques, and Neves, in an astonishingly quick argument we give below. The dimensional restrictions come from the regularity theory for Almgren–Pitts minmax. Yangyang Li proved a result along these lines for generic metrics in higher dimensions allowing for some singularities.

Now, for each nontrivial sweepout (i.e. nonzero width) we get a minimal surface. To use minmax to attack Yau's conjecture then there are two (1.5?) issues:

(1) Produce infinitely many distinct (or at least seemingly distinct) sweepouts, which are candidates to give rise to infinitely many minimal surfaces. The involves a little bit of topology but is not too bad.
(2) Actually show the minimal surfaces from item (1) are geometrically distinct i.e. not the same finite set of ones but appearing with different multiplicities. There are a couple different ways to go about this:
    (a) Show a multiplicity one theorem. A result along these lines was shown by Xin Zhou which we'll hopefully get to at the end of these notes. This is the most direct approach.
    (b) Use the Weyl law for the volume spectrum ala Gromov and Guth (actually as we'll see below one can already get a good result using a bit weaker fact). These give area bounds on the minimal surfaces one finds from p dimensional sweepouts which are sublinear in p and a pidgeonhole type argument rules out getting the same old minimal surfaces with multiplicity. This is historically the oldest approach.

Let us start off then by discussing how to deal with item (1), which is the main focus in this section. To do this we need to talk a little bit about currents, which are another central object in geometric measure theory alongside varifolds: in practice, restricting to good subsets, these are spaces of varifolds with extra algebraic structure (we can talk about boundaries and cycles) in some sense which can often be useful. Spewing out some definitions:

- Denote by $\mathcal{D}^k(\mathbb{R}^J)$ the set of smooth k-forms of $\mathbb{R}^J$ with compact support, and given an element $\omega \in \mathcal{D}^k(\mathbb{R}^J)$ define $|\omega| = \sup_{x\in\mathbb{R}^j}\{\langle \omega(x), \omega(x)\rangle^{1/2}\}$ (this pairing is via the Hodge star).
- A k-current T is a continuous linear functional on $\mathcal{D}^k(\mathbb{R}^J)$. Its boundary ∂T is a $k-1$ current that is defined as $\partial T(\phi) = T(\partial\phi)$ for $\phi \in \mathcal{D}^{k-1}(\mathbb{R}^J)$. From this definition note that $\partial^2 T = 0$.
- Now we start to restrict to spaces of currents which are "dressed up" varifolds: we say that T is an integer multiplicity k-current if it can be expressed as

$$T(\phi) = \int_S \langle \phi(x), \tau(x)\rangle \theta(x) d\mathcal{H}^k,\ \ \phi \in \mathcal{D}^k(\mathbb{R}^J) \tag{3.1}$$

where S is a rectifiable varifold, θ is $\mathcal{H}^k$–integrable $\mathbb{N}$ valued function, and τ is a k form so that for all $x \in S_*$ (the regular part of S), $\tau(x)$ is a volume form for T_xS.

- The space of k–currents T such that both T and ∂T are integrable multiplicity currents with finite mass and support contained in M is denoted by $\mathbf{I}_k(M)$ ad is called the **space of integral k–currents**. The space of k–cycles $\mathcal{Z}_k(M)$ is the space of those $T \in \mathbf{I}_k(M)$ so that $T = \partial Q$ for some $Q \in \mathbf{I}_{k+1}(M)$ and morally corresponds to the support being the boundary of a $k+1$ submanifold.
- An important subset of $I_k(M), \mathcal{Z}_k(M)$ is the space of mod 2 integral k-currents and k-cycles, denoted by $\mathbf{I}_k(M;\mathbb{Z}_2)$ and $\mathcal{Z}_k(M;\mathbb{Z}_2)$ respectively. Here we say that $T \equiv S$ when $T - S = 2Q$ for some $Q \in \mathbf{I}_k(M)$. We will use it shortly below.

Now there are a few different norms/topologies one can put on the space $I_k(M)$ that have pros and cons and pathologies. Since we already discussed Simon–Smith minmax at length I don't want to get into the weeds here too much but in Almgren–Pitts instead of considering the "nearly smooth" sweepouts we did before one considers much more general continuous families of currents in the mass topology. The "real" Almgren–Pitts minmax theorem then is:

Theorem 3.2 (Almgren–Pitts, roughly stated)**.** *Suppose that $2 \leq n \leq b$, X is some m dimensional simplicial complex, and suppose $\Pi \in [X, \mathcal{Z}_n(M)]$ with the width $L(\Pi) > 0$ (switching to Marques and Neves' notation). Then there is a stationary integral varifold Σ, whose support is a (potentially disconnected) smooth embedded hypersurface with mass $||\Sigma||(M) = L(\Pi)$.*

Note that because we are considering maps into cycles (which are boundaries – this is perhaps a bit confusing compared to homology theory but is what people call things in this field), the minmax limits one finds will often be topologically trivial because, if the limit is connected (as in the Ricci positive case) and with multiplicity one, it must be nullhomologous. Now with this new language introduced we can finally get back to discussing item (1) way above, or to produce "many sweepouts." Let $f : M \to [0,1]$ be a Morse function and consider the map $\hat{\Phi} : \mathbb{R}P^\infty \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ given by

$$\hat{\Phi}([a_0 : a_1 : \cdots : a_k : 0 : 0 : \cdots]) = \partial\{x \in M \mid a_0 + a_1 + f(x) + \cdots + a_k f(x)^k \leq 0\} \quad (3.2)$$

This is a well defined map because we are considering mod 2 cycles (and as a technical remark is continuous in the flat topology, which can be homotoped to a continuous one in the mass topology). The big fact then is the following, originally due to Almgren:

Theorem 3.3. *$\hat{\Phi} : \mathbb{R}P^\infty \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is a weak homotopy equivalence.*

Proof: [sketch] The basic idea is that there is a very natural and simple to understand covering space for the space of cycles: consider the map $\partial : \mathbf{I}_{n+1}(M;\mathbb{Z}_2) \to \mathcal{Z}_n(M;\mathbb{Z}_2)$, which we first claim is 2 to 1. To convince ourselves suppose that the support Σ is a smooth boundaryless hypersurface of M which is a cycle in the sense of currents so that it bounds a domain U. Now because we are considering $\mathbb{Z}_2$ coefficients in the definition of current we have $\theta = 1$, and the preimage in $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ comes from one of the two domains $U, \overline{U}^c$ (of course to be more precise we need to discuss the topology on the space of currents we are using, but let's not here).

Next, we claim that $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is contractible. To see this, consider the map $H : [0,1] \times \mathbf{I}_{n+1}(M;\mathbb{Z}_2) \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ by setting $H(t,U) = U \cap \{x \in M \mid f(x) \le t\}$ (this is slightly wrong but I couldn't find the correct latex symbol for interior multiplication). This is continuous and retracts domains to the empty set. Note that because ∂ is continuous it implies that $\mathcal{Z}_n(M;\mathbb{Z}_2)$ is path connected.

Secondly, we claim that ∂ is a covering map. Intuitively this is because the map ∂ is a 2-1 mapping in the topology we use and the different domains bounded by an element in $\mathcal{Z}_n(M;\mathbb{Z}_2)$ are far apart from each other, so we can safely vary things continuously in $\mathcal{Z}_n(M;\mathbb{Z}_2)$ and get two different open sets in $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$.

Now consider a continuous map $\Psi : S^k \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ for $k \ge 2$. because ∂ is a covering map and S^k for $k \ge 2$ is simply connected this lifts to a map $\tilde{\Psi} : S^k \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$, and because this space is contractible $\tilde{\Psi}$ and hence Ψ can be homotoped to a constant map giving $\pi_k(\mathcal{Z}_n(M;\mathbb{Z}_2))$ is trivial for $k \ge 2$. Similarly because ∂ is a 2-fold covering map we get that $\pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2)) = \mathbb{Z}_2$.

So, the homotopy groups of $\mathbb{R}P^\infty$ and $\mathcal{Z}_n(M;\mathbb{Z}_2)$ agree and we just want to check last that the map above exhibits this. Considering the curve

$$t \to [\cos(\pi t) : \sin(\pi t) : 0 :, \cdots] \tag{3.3}$$

which generates $\pi_1(\mathbb{R}P^\infty)$ The image of this loop under $\hat{\Phi}$ is the set $\partial\{f \le -\cot(\pi t)\}$, $0 \le t \le 1$. This lifts to a curve from 0 to M (-0 if you will) in $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ so is homotopically nontrivial for the same reason the loop from the quotient of the path from the north to south pole in $\mathbb{R}P^2$ is. Because these spaces have trivial higher homotopy groups this completes the argument. □

The proof above is actually due to Marques and Neves, Almgren's is apparently more difficult. What he shows in his thesis is the following:

Theorem 3.4 (Almgren, PhD thesis)**.** *For any smooth closed Riemannian manifold M and any nonnegative integers k, m, we have*

$$\pi_k(\mathcal{Z}_m(M)) \simeq H_{k+m}(M) \tag{3.4}$$

This implies the result above for $\mathbb{Z}_2$ coefficients when $m = n-1$. Now, this is quite inspirational because the space $\mathbb{R}P^\infty$ has cohomology ring $H^*(\mathbb{R}P^\infty, \mathbb{Z}_2) = \mathbb{Z}_2[\lambda]$. With

Morse theory in mind (using the area functional) without being too careful this should appear to imply there are infinitely many stationary cycles, considering our discussion of Lusternik-Schnirelmann theory from the introduction. Proceeding more carefully, the result above and the Hurewiscz theorem imply that

$$H^1(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \overline{\lambda}\} \tag{3.5}$$

here we call $\overline{\lambda}$ the fundamental cohomology class (indeed, we already saw this appear in Priya Kevari's talk).

We use this to define nontrivial sweepouts. Letting $k \in \mathbb{N}$ and X a finite dimensional cubical subcomplex an m-dimensional cube I^m, a continuous map $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is called a k–sweepout if $\lambda = \Phi^*(\overline{\lambda}) \in H^1(X, \mathbb{Z}_2)$ satisfies:

$$\lambda^k = \lambda \smile \cdots \smile \lambda \neq 0 \in H^k(X, \mathbb{Z}_2)$$

The set of all k–sweepouts is denoted by $\mathcal{P}_k$ (note I'm being imprecise about what topology on the space of currents I'm using now as usual). Using that $\mathbb{R}P^k \subset \mathbb{R}P^\infty$ in the obvious way its not hard to see that $\mathcal{P}_k$ is nonempty, and the Federer–Flemming isoperimetric inequality implies that these are nontrivial in these sense that the width is positive: basically if the width were zero, then the FF isoperimetric inequality gives a continuous family of $n+1$ currents we could retract across to see the family was topologically trivial.

These sweepouts give corresponding saturations with which we may run the minmax machine, from which we should get infinitely many stationary cycles. But there are issues, for instance as we mentioned they might not be geometrically distinct. We discuss this next.

4. Getting geometrically distinct minimax limits: The Weyl law approach

The k width of a Riemannian manifold (M, g) is the width varying over families in the family $\mathcal{P}_k$ defined above:

$$\omega_k(M, g) = \inf_{\Phi \in \mathcal{P}_k} \sup_{x \in \mathrm{dmn}(\Phi)} \mathrm{area}(\Phi(x)) \tag{4.1}$$

Because any k sweepout is an ℓ sweepout for $\ell < k$ and we are taking an infimum above the sequence of numbers $\{\omega_k(M, g)\}$ is non–decreasing and is called the volume spectrum of (M, g).

First let's see why its actually fair to call this sequence as a "spectrum," where a spectrum we usually think of as the eigenvalues of some self adjoint operator. Considering perhaps the most familiar one of them all, the Laplacian (on a compact manifold) with eigenvalues

$0 = \lambda_0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots$, we recall the Rayleigh quotient characterization of the k-th eigenvalue:

$$\lambda_k = \inf_{(k+1)-\text{plane } P \subset W^{1,2}(M)} \max_{f \in P - \{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2} \tag{4.2}$$

This is easy to convince yourself of using the spectral theorem, which gives a orthonormal basis of L^2 by smooth eigenfunctions $f_1, f_2, \ldots$, as follows. λ_0 of course corresponds to the 1-plane (i.e. line) given by the constants. Next consider a 2–plane P and let's suppose it saturates the characterization for λ_1; such a plane exists by Relich–Kondrachov compactness. Also suppose that we have an orthonormal basis of P given by $\{1, h\}$ – it will be clear from the following that we can reduce to this case. Since h is orthogonal to 1 we must be able to write it as $\sum_{i=1}^{\infty} c_i f_i$. Plugging h into the quotient of course the denominator above is normalized to be 1, and the numerator using integration by parts and the definition of f is $\sum_{i=1}^{\infty} c_i^2 \lambda_i$. Since the c_i^2 themselves must sum up to one because h is normalized, we see we must have $c_1 = 1$ and the rest be zero because $\lambda_i > \lambda_1$ or so that $h = f_1$ because we are taking an infimum/by the choice of P. Considering $f = c_0 + c_1 f_1 \in P$, where by scale invariance of the quotient without loss of generality $c_0^2 + c_1^2 = 1$, by similar reasoning if f maximizes the quotient above $c_0 = 0$ and $c_1 = 1$, so that the right hand side above over 2-planes is λ_1.

The case for higher λ_k is hopefully clear at this point – the point being that we have in analogy to ω_k a minmax characterization of λ_k (which is surely ancient knowledge, we are working backwards here). And if you need more convincing of the analogy, note that the spectral theorem also implies that the projectivization of $W^{1,2}(\Omega)$ is $\mathbb{R}P^{\infty}$, similar to the result above on $\mathcal{Z}_n(M, \mathbb{Z}_2)$!

Now, an interesting classical fact about eigenvalues of the Laplacian is that we have a understanding in a sense of how they are distributed – maybe as some inspiration think of the following as somehow a PDE version of the prime number theorem. Denoting by $N_\Omega(T) = \#\{\lambda_k \leq T\}$ for a domain in $\mathbb{R}^n$, one may be interested in the asymptotics of N as $T \to \infty$ where here we are considering eigenvalues to the Laplacian with Dirichlet boundary condition (i.e. vanishing along boundary). As some inspiration consider first the $1D$ case on $\Omega = [0, a]$, where we are interested in solutions to the equation:

$$-u'' = \lambda u, u(0) = u(a) = 0 \tag{4.3}$$

Solving this ODE (using an ansatz) we see we get a basis of L^2_0 by solutions given by $u_k = \sin(\frac{k\pi}{a} x)$ for the eigenvalues $\lambda_k = (\frac{k\pi}{a})^2$ (and since these are a basis of solutions, its all of them!). Then we see that

$$N(T) = \#\{k \in \mathbb{N} \mid \lambda_k < T\} = \max\{k \in \mathbb{N} \mid k < \frac{a\sqrt{T}}{\pi}\} \sim \frac{a\sqrt{T}}{\pi} = \frac{\text{area}(\Omega)}{\pi}\sqrt{T} \tag{4.4}$$

With the middle equality because the eigenvalues are all simple (i.e. with multiplicity one). Similarly using separation of variables for domains $\Omega = [0, a] \times [0, b]$ a basis of eigenfunctions is $u_{j,k} = \sin(\frac{j\pi}{a}x)\sin(\frac{k\pi}{b}y)$ and using this one can see $N(T) \sim \frac{abT}{4\pi} = \frac{\text{area}(\Omega)}{4\pi}T$. One can of course work things out explicitly for rectangles in $\mathbb{R}^n$ for general n along the same lines. With this in mind Lorentz conjectured, and Weyl proved, the Weyl law:

Theorem 4.1 (Weyl's law)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. then the counting function $N(\Omega; T)$ satisfies:*

$$\lim_{T\to\infty} \frac{N(\Omega; T)}{T^{n/2}} = \frac{\omega_n}{(2\pi)^n} vol(\Omega) \tag{4.5}$$

The proof is not bad and the proof of the corresponding Weyl law for the volume spectrum takes some inspiration from it, so let's discuss it. Indeed, the Weyl law is true for solid rectangles in $\mathbb{R}^n$ and this is actually quite important in the argument. Another important observation to show this is the following:

Lemma 4.2 (domain monotonicity for eigenvalues)**.** *Let V be a bounded domain of $\mathbb{R}^n$ and $U_1, U_2, \ldots, U_\ell \subset\subset V$ be piecewise smooth subdomains with pairwise disjoint closures. Denoting by $\mu_{i,k}$ the k–th eigenvalue for U_i we have*

$$\lambda_k \leq \mu_{i,k} \tag{4.6}$$

for any i. As a matter of fact:

$$N(U_1; T) + \ldots + N(U_\ell; T) \leq N(V; T) \tag{4.7}$$

Proof: [sketch] To show that $\lambda_k \leq \mu_{i,k}$, note we can take a solution to the Dirichlet problem in U_i and extend it by zero (at least weakly, working in Sobolev spaces) to get a solution for the Dirichlet problem on all of V. Thus $(k+1)$-planes in $W^{1,2}(U_i)$ give rise to $(k+1)$-planes in $W^{1,2}(V)$ so that $W^{1,2}(V)$ has at least as many "competitor planes" in the Rayleigh quotient characterization of λ_k. Since an infimum is being taken (taking an infimum over a larger set could give a smaller number) this gives the first assertion.

Now for the second assertion, which is a "superadditivity" of sorts of the counting function. To get started note that if we took ℓ eigenfunctions $u_1, \ldots, u_\ell$ of a U_i they would give rise to ℓ eigenfunctions of V. To see this first you extend them to all of V to get an ℓ plane L of $W^{1,2}(V)$ (the dimension of the plane doesn't go down, because they are orthogonal functions which stay orthogonal under extension). By the definition of L we see the Laplacian restricted to L maps into L (in the weak sense that if B is the bilinear form associated to Δ then $B[u, v] = 0$ if $u \in L$ and $v \in L^\perp$), so by the spectral theorem there is a basis for this plane by Dirichlet eigenfunctions $v_i, \ldots, v_\ell$ for the domain V. By using the extended functions as competitors like in the paragraph above, the corresponding eigenvalues $\lambda(v_1), \ldots, \lambda(v_\ell)$ of $v_1, \ldots, v_\ell$ are bounded above by $\max\{\lambda(u_1), \ldots, \lambda(u_\ell)\}$, so if this is bounded above by T so are $\lambda(v_1), \ldots, \lambda(v_\ell)$. This gives that $N(U_i; T) \leq N(V; T)$ for each i from 1 to ℓ.

Now, to get what we really want note by the same reasoning $N(\cup_i U_i; T) \leq N(V; T)$ (we didn't make any sort of stipulation about the U_i being connected). Now because the U_i are pairwise disjoint eigenfuntions in disjoint U_I, U_j are linearly independent in $W^{1,2}(\cup_i U_i)$, giving that $N(\cup_i U_i; T) = N(U_1; T) + \ldots + N(U_\ell; T)$. □

Proof: [sketch of Weyl's law] Choose pairwise solid rectangles $U_i \subset\subset \Omega$ which fill Ω up to error ϵ. By the lemma above we have:

$$\frac{1}{T^{n/2}} \sum_i N(U_i, T) \leq \frac{N(\Omega; T)}{T^{n/2}} \tag{4.8}$$

Because Weyl's law for rectangles holds, we get

$$\frac{\omega_n}{(2\pi)^n}(\mathrm{vol}(\Omega) - \epsilon) \leq \liminf \frac{N(\Omega; T)}{T^{n/2}} \tag{4.9}$$

We are taking liminfs here because we don't even know if the limit exists yet on general domains! For the opposite inequality put Ω into a huge rectangle R, and the fill $R \setminus \overline{\Omega}$ by smaller rectangles $R_1, \ldots, R_\ell$ again up to error ϵ. We have that

$$\frac{N(\Omega; T)}{T^{n/2}} + \frac{1}{T^{n/2}} \sum_i N(R_i, T) \leq \frac{R; T)}{T^{n/2}} \tag{4.10}$$

Bringing the sum of the smaller rectangles to the RHS and using again that the Weyl law for rectangles is true, we get

$$\limsup \frac{N(\Omega; T)}{T^{n/2}} \leq \frac{\omega_n}{(2\pi)^n}(\mathrm{vol}(\Omega) + \epsilon) \tag{4.11}$$

Since $\epsilon > 0$ was arbitrary, we get the claim. □

Of course there are generalizations of this result, including to smooth manifolds, and the above proof is not the only way to proceed. One interesting implication of Weyl's law is that you can hear the volume of the drum, but as you probably have at least heard about you can't in general hear the shape. Getting back to the volume spectrum:

Theorem 4.3 (Weyl law for the volume spectrum, by Marques, Neves, and Liukomovich)**.** *There exists a constant $a(n) > 0$ such that, for every compact Riemannian manifold (M^{n+1}, g) with possibly empty boundary, we have*

$$\lim_{k \to \infty} \omega_k(M) k^{-\frac{1}{n+1}} = a(n) vol(M)^{\frac{n}{n+1}} \tag{4.12}$$

As stated this isn't exactly in the spirit of the classical Weyl law, of course, because it only says something about the asymptotic relationship of the k–widths (eigenvalues) to the volume of M. A bit below we deduce from it and some intermediate reductions an inequality for counting functions. A weaker version of this result was first observed by Gromov and Guth:

Theorem 4.4. *There is a constant $C = C(M,g) > 0$ so that for all $k \in \mathbb{N}$*

$$\frac{1}{C}k^{\frac{1}{n+1}} \leq \omega_k(M) \leq Ck^{\frac{1}{n+1}} \tag{4.13}$$

An important part of the proof of this Gromov–Guth result is, or at least seems to be highly related to (following Guth), the deformation theorem which says that we can deform a current to (the boundary of) a cubical subdivision of M with controlled gain/loss in mass. This is a good result to at least be aware of and is how you prove the compactness theorem and isoperimetric inequality of Federer and Flemming. Then in a nutshell, given a suitable element of $\mathcal{P}_k$ from above, one can deform its elements to a nearby surface which certainly satisfies one of the bounds in a controlled way – this isn't horribly hard but perhaps it is a bit much to go into depth here. Maybe I can at least give some intuition for why the bounds should grow in k, anyway: one can see that in a nontrivial k parameter sweepout at least one of the elements must pass through k points, so in a rough way such sweepouts must contain a high area element.

For the Weyl law of the volume spectrum, at a very high level, one can then observe from this that the Weyl law for the volume spectrum holds for at least *some* constant $a(n)$ on small Euclidean cubes by the result above, and then use a covering argument a bit similar to the proof of the classical Weyl law for general closed manifolds to get the Weyl law for volume spectrum. The stand-in for lemma 4.2 above is the following:

Lemma 4.5 (Lusternik-Schnirelman superadditivity)**.** *Let $U_0, \ldots, U_\ell \subset \Omega$ be connected piecewise disjoint Lipschitz domains contained in a domain Ω. Then given integers $k_i + \cdots + k_\ell \leq k$, we have*

$$\sum_i \omega_{k_i}(U_i) \leq \omega_k(\Omega) \tag{4.14}$$

Moving on with our lives, we next discuss a couple prepatory facts for proving the Yau conjecture (in some cases).

Lemma 4.6 (equality of widths)**.** *Fixing k and denoting the saturation of a sweepout $\Phi \in \mathcal{P}_k$ by $[\Phi]$, without loss of generality the minmax width $L([\Phi])$ of $[\Phi]$ equals the k-width ω_k.*

Proof: [sketch] Note that by definition of minmax width and that if $\Phi \in \mathcal{P}_k$ every element of $[\Phi]$ is too so that $\omega_k(M) \leq L([\Phi])$. If the inequality is strict, we can find a sequence of k sweepouts Φ_i such that $L([\Phi_1]) > L([\Phi_2]) > L([\Phi_3]) > ... \to \omega_k(M)$ and for each of these sweepouts we get closed embedded minimal surfaces V_k. We must not keep obtaining the same finite number of minimal surfaces with multiplicity because the sequence of minmax widths above is decreasing and bounded from above. □

Lemma 4.7 (Lusternik-Schnirelmann theory application). *Without loss of generality, the sequence $\{\omega_k(M)\}$ is strictly increasing.*

Proof: [sketch] Again the claim is that if $\omega_k(M) = \omega_{k+1}(M)$ for some k we must have infinitely many minimal surfaces. Suppose by contradiction that the set $\mathcal{T}$ of minimal surfaces with area bounded above by $\omega_{k+1}(M)$ is finite. Choosing a sequence of $\mathcal{P}_{k+1}$ sweepouts Φ_i whose maximal slice is approaching $\omega_k(M) = \omega_{k+1}(M)$ parameterized over X_i, consider the following subsets:

$$A_i = \{x \in X_i \mid d(\Phi_i(x), \mathcal{T}) < \epsilon\} \tag{4.15}$$

and

$$B_i = \{x \in X_i \mid d(\Phi_i(x), \mathcal{T}) > \epsilon/2\} \tag{4.16}$$

(As I've gotten into the habit of in these notes, I'm ignoring what topology we are considering here.) If ϵ is small enough the set A_i is a finite union of contractible sets so that the fundamental class $\overline{\lambda}$, vanishes on it. Now here is the "Lusternik–Schnirelmann" bit: If $\overline{\lambda}^k$ vanished on B_i, then $\overline{\lambda}^{k+1}$ vanishes on $X_i = A_i \cup B_i$ – the proof of this can be deduced from the proof of theorem 1.2 above. This is a contradiction because the Φ_i belong to $\mathcal{P}_{k+1}$.

So we see that the sweepouts Φ_i also belong to $\mathcal{P}_k$ and in particular this implies $W \leq \omega_k(M)$, where $W = \inf_i \sup_{x \in B_i} ||\Phi_i(x)||$. One the other hand because these sweepouts detect $\overline{\lambda}^k$ we get that in fact $W = \omega_k(M)$. Any minmax limits from these sweepouts must be away from the set $\mathcal{T}$, so that the regularity theory must break down somewhere: in particular, they aren't almost minimizing in annuli. Using the subsequent area decreasing deformations from this we see in fact $\omega_k(M) < \omega_{k+1}(M)$, giving a contradiction. □

4.1. **Yau's conjecture in most cases.** With these in hand let's discuss how Marques and Neves showed the following, which was their first major result in the direction of Yau's conjecture:

Theorem 4.8 (Yau conjecture in Ric > 0). *Let (M, g) be a compact Riemannian manifold of dimension $(n+1)$, $2 \leq n \leq 6$, and positive Ricci curvature. Then M contains an infinite number of distinct smooth, closed, embedded minimal hypersurfaces.*

Proof: [yes, a sketch!] Denote by $\mathcal{L}$ the set of embedded closed minimal hypersurfaces in M and suppose it is finite. Now, by the minmax theorem and the first fact above we have that for each k ther exists a stationary varifold V_k so that $\omega_k(M) = ||V_k||$, where the support of V_k is a smooth closed embedded minimal hypersurface in $\mathcal{L}$ possibly with multiplicity. Since the elements of $\mathcal{L}$ are all minimally embedded and M has positive Ricci curvature, by the Frankel property we get that $V_k = n_k\Sigma_k$ for a single element $\Sigma_k \in \mathcal{L}$.

In the following we proceed by a pigeonhole argument, more or less. By lemma 4.7 above we have $||V_k|| < ||V_{k+1}||$ for all $k \in \mathbb{N}$ and by the Weyl law (or really here we just need the Gromov–Guth bounds) $n_k|\Sigma_k| = \omega_k(M) \leq Ck^{\frac{1}{n+1}}$. Writing N for the number of elements of $\mathcal{L}$, and δ as an area lower bound for elements $\Sigma \in \mathcal{L}$ (such a number exists and is strictly positive since $\mathcal{L}$ is finite) we get that the number of possible areas of minimal surfaces coming from ℓ–sweepouts for $\ell \leq k$, which more compactly is the number

$$\#\{a = m|\Sigma| \mid m \in \mathbb{N}, \Sigma \in \mathcal{L}, m|\Sigma| \leq Ck^{\frac{1}{n+1}}\} \tag{4.17}$$

is bounded above by (number of elements of $\mathcal{L}$)$\times\left(\frac{Ck^{\frac{1}{n+1}}}{\text{the smallest potential area for an element of } \mathcal{L}}\right) = N\frac{Ck^{\frac{1}{n+1}}}{\delta}$. In particular this bounds the number of stationary currents coming from ℓ sweepouts for ℓ ranging from 1 to k. On the other hand since without loss of generality $\omega_\ell \neq \omega_{\ell+1}$ for any ℓ we have that there are k different stationary currents coming from these, where here we are considering stationary currents with different multiplicities as distinct. In particular we must have

$$k \leq N\frac{Ck^{\frac{1}{n+1}}}{\delta} \tag{4.18}$$

Because $k^{\frac{1}{n+1}}$ grows sublinearly we get a contradiction, so $\mathcal{L}$ must not be a finite set. □
Using the full strength of the Weyl law for the volume spectrum, Marques and Neves along with Irie were able to prove the conjecture in wide generality:

Theorem 4.9 (Yau's conjecture for generic metrics, Irie-Marques-Neves)**.** *Let M^{n+1} be a closed manifold of dimension $(n+1)$, with $3 \leq n+1 \leq 7$. Then for a C^∞ generic Riemannian metric g on M, the union of all closed, smooth, embedded minimal hypersurfaces is dense.*

In fact with Song, Marques and Neves later went on to show that the space of minimal surfaces are equidistributed in a strong sense using again the Weyl law for generic metrics, but we won't discuss this here (at least let). The proof of the fact above is short and sweet:
Proof: [sketch] Denoting by $\mathcal{M}$ the space of metrics on M, let $\mathcal{S}(g)$ denote the set of all connected, smooth, embedded minimal hypersurfaces with respect to g. For an open set $U \subset M$ set

$$\mathcal{M}(U) = \{g \in \mathcal{M} \mid \exists \Sigma \in \mathcal{S}(g) \text{ with } \Sigma \cap U \neq \emptyset\} \tag{4.19}$$

The set is open by the inverse function theorem. If one can show it is dense in $\mathcal{M}$ for any open set U the result follows: because M is second countable we can consider a countable basis $\{U_i\}$ of M and by the Baire category theorem the set $\cap_i \mathcal{M}_{U_i}$ is dense in $\mathcal{M}$. This implies that for any open set $V \subset M$ endowed with a metric from the intersection above we can find a minimal surface Σ intersecting it. So, pick some proper open set V, we get a minimal surface Σ. Picking another open set V_1 properly contained in $M \setminus \Sigma$ we get another distinct minimal surface Σ_1... continuing this process gives a countably infinite number of distinct minimal surfaces Σ_i.

Now, consider an arbitrary $g \in \mathcal{M}$ and let $B \subset \mathcal{M}$ be a neighborhood of g. By White's bumpy metric theorem, there exists $g' \in B$ such that ever minimal surfaces in (M, g') is nondengenerate. By the compactness theorem of Sharp the set of minimal surfaces $\Sigma \subset (M, g')$ of bounded area and index is finite (by compactness we can cover the set with finitely many open sets, and in each open set there must be finitely many elements or else we could extract a Jacobi field), implying that $\mathcal{S}(g')$ is countable and that the set $\mathcal{C} := \{\mathrm{vol}_{g'}(V) \mid V \text{ is a smooth embedded cycle}\}$ is also countable.

With this in mind pick an open set $U \subset M$, h a nonnegative (and nonzero) smooth function supported in U, and consider the deformation $g'(t) = (1 + th)g'$ on the interval $[0, t_0]$ for which $g'(t) \in B$. Because h is nonnegative and nonzero, we see that $\mathrm{vol}(M, g'(t)) > \mathrm{vol}(M, g')$ so by the Weyl law there exists k such that $\omega_k(M, g'(t_0)) > \omega_k(M, g')$.

Now suppose on the contrary that $B \cap \mathcal{M}_U = \emptyset$. Then for every $t \in [0, t_0]$ every closed, smooth, embedded minimal hypersurface in $(M, g'(t))$ is contained in $M \setminus U$. Since $g' = g$ outside U we get that (without loss of generality, from work above) the number ω_k belongs to to the set $\mathcal{C}$ from the paragraph above, which is countable. But one can see that ω_k depends continuously on the metric, which since $\omega_k(M, g'(t_0)) > \omega_k(M, g')$ gives a contradiction because the value would have to jump for some $t \in [0, t_0]$. Hence $B \cap \mathcal{M}_U \neq \emptyset$ implying, since g, B were arbitrary, that $\mathcal{M}_U$ is dense in $\mathcal{M}$ as needed. □

To sum up the proof above in one sentence the bumpiness of g the k-widths were discrete, but the Weyl law says (roughly, asymptotically) that they vary continuously in some sense.

5. Xin Zhou's multiplicity one theorem for minmax

After the reductions that the sequence $\{\omega_k(M)\}$ of k-widths is strictly increasing and actually correspond to minmax widths for appropriate sweepouts (see ithe lemmas preceeding the sketch of theorem 4.8), we note that the *only* obstruction to proving Yau's conjecture is that the minmax limits achieving the widths may the same finite set just with differing multiplicities. The proofs above ruled these issues out, at a very high level, by a pidgeonhole argument and a continuity argument respectively using the Weyl law for the volume spectrum but perhaps the most straightforward thing to do is simply to rule this possibility out.

This is natural, at least in the bumpy setting, which is generic by White, because intuitively if a minimal surface Σ appears as a minmax limit with multiplicity greater than one, and supposing the convergence is graphical away from a small set (which will be the case below) the difference of these graphs should give a positive solution to the Jacobi equation which implies that Σ has a nontrivial Jacobi field, giving a contradiction to the bumpy assumption. Indeed, an elaboration of this idea is at the core of the argument below. Writing the minmax minmal surfaces achieving the k-width by $\sum_{i=1}^{\ell_k} m_i^k \Sigma_i^k$ Zhou's theorem, then, is the following:

Theorem 5.1 (Multiplicity one for minmax). *Given a closed manifold M^{n+1} of dimension $3 \leq (n+1) \leq 7$ with a bumpy metric g, the minmax minimal hypersurfaces $\{\Sigma_i^k \mid k \in \mathbb{N}, i = 1, \cdots, \ell_k\}$ achieving ω_k are all two-sided, have multiplicity one, and index bounded by k.*

A central technical tool in the proof is the development, by Zhou himself and Zhu, of the minmax theory for surfaces of prescribed mean curvature. Denoting by $\mathcal{C}(M)$ the set of Caccioppoli sets in M^{n+1} (i.e. sets in M of locally bounded parameter i.e. defined in terms of characteristic sets of bounded variation and give rise to currents) they define the $\mathcal{A}^h$ functional as:

$$\mathcal{A}^h(\Omega) = \mathcal{H}^n(\partial\Omega) - \int_\Omega h d\mathcal{H}^{n+1} \tag{5.1}$$

for $\Omega \in \mathcal{C}(M)$ and, say, smooth functions $h : M \to \mathbb{R}$. When $h = 0$ this reduces to the area functional, clearly. The first variation formula with respect to a C^1 vectorfield X is

$$\delta\mathcal{A}^h \mid_\Omega (X) = \int_{\partial\Omega} \mathrm{div}_{\partial\Omega} X - h\langle X, \nu\rangle \tag{5.2}$$

When $\partial\Omega$ is smooth, of course the first term is just $H\langle X, \nu\rangle$ giving that sufficiently regular critical points of $\mathcal{A}^h$ should satisfy $H = h$. You can also consider the second variation of $\mathcal{A}^h$ and discuss the index of a critical point as in the classical minimal case, which is actually important for the task at hand as we'll see.

Finding a surface Σ so that $H = h$ for a given h is called the prescribed mean curvature (PMC) problem. Naturally there are some analytical differences between the PMC problem and properties of such surfaces Σ and the theory of minimal surfaces, but it turns out that one can use minmax to find PMC surfaces, where below X is a cubical complex of dimension k and Z is a cubical subcomplex of X:

Theorem 5.2 (Zhou, Zhu minmax theorem for PMC, roughly stated). *Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$ and $h \in \mathcal{S}(g)$ (this is a dense subset of $C^\infty(M)$ consisting of special Morse functions) which satisfies $\int_M h \geq 0$.*

Then given a sweepout Φ *(by Caccioppoli sets) and an associated* (X, Z)*-homotopy class* Π*, suppose*

$$L^h(\Pi) > \max_{z \in Z} \mathcal{A}^h(\Phi)(z)) \tag{5.3}$$

Then there exists a nontrivial smooth, closed, almost embedded hypersurface $\Sigma \subset M$ *of prescribed mean curvature* h *with multiplicity one achieving the width of* Π*.*

L^h is the obvious defintion of width following the classical minmax definition. Almost embeddedness here means that Σ may touch itself tangentially but not transversely; this is a possibility because PMC surfaces generally don't satisfy a strong maximum principle. The condition $\int_M h$ implies that $\mathcal{A}^h(M) \leq 0$ – there exists positive width sweepouts so that one obtains nontrivial PMC surfaces and if $\int_M h \leq 0$ one can just considering $-h$ and flip the orientation of the normal on the resulting surface. The overall minmax scheme in this setting mirrors the classical case (although not trivially, the result above is in Inventiones) so for the sake of brevity we skip discussing it here.

One especially interesting point for us though is that the PMC surfaces above appear with multiplicity one; as some hand wavy justification for why this makes sense supposing that $h > 0$ in a neighborhood where the convergence is with high (≥ 2) multiplicity by disjoint sheets then each sheet in the convergence should have the same orientation, since on them $H \sim h > 0$ and by Allard regularity are basically parallel to the limit surface. On the other hand since the surfaces come from the boundaries of Caccioppoli sets the orientations between sheets should alternate, contradicting this.

With some optimism this is inspirational, for instance because one might hope to approximate the area functional $\mathcal{A}^0$ by $\mathcal{A}^{h_k}$ for $h_k \to 0$ and realize (minimal) minmax limits V by the multiplicity one minmax surfaces V_i for $\mathcal{A}^{h_k}$, and hopefully this would rub off on V. This isn't quite how Zhou proceeds (and maybe not how he thought about it at all) but indeed he approximates the area functional roughly as such.

Proof: [sketch of theorem 5.1] Considering h admissible in the PMC minmax theorem above, it turns out that ϵh will also be in $\mathcal{S}(g)$. Considering a k–parameter sweepout Φ, the family Σ_ϵ from the PMC minmax theorem will have uniformly bounded area and index, so that one can extract a converging subsequence ala Sharp's compactness theorem as $\epsilon \to 0$ (although this is in the PMC setting!) to a surface Σ_∞. This limit surface Σ_∞ must be minimal, have index bounded by k, and have area agreeing with the width of Φ – in fact one can see that it can be taken to be a minimal surface achieving the k–width.

The convergence $\Sigma_{\epsilon h} \to \Sigma_\infty$ will also be (multi-)graphical away from a finite set of points $\mathcal{Y}$, from the essentially from the regularity theory for minmax, and as we alluded too above (as some justification of the multiplicity one conjecture) Simon's method can be applied. By this we mean, writing $\Sigma_{\epsilon h}$ for $\epsilon << 1$ over $\Sigma \setminus \mathcal{Y}$ these surfaces can be written as graphs of functions $u_\epsilon^1 \leq \cdots u_\epsilon^m$, and as is well known the difference of the top and bottom graphs will nearly satisfy an equation in terms of the Jacobi operator L_{Σ_∞}. If the number m is odd, these graphs have the same orientation and one finds they satisfy the equation:

$$L_{\Sigma_\infty}(u_\epsilon^m - u_\epsilon^1) + o(u_\epsilon^m - u_\epsilon^1) = \epsilon \partial_\nu h \cdot (u_\epsilon^m - u_\epsilon^1) \tag{5.4}$$

Rescaling by the difference of the u^i, letting $\epsilon \to 0$ and applying a removable singularity theorem one gets a positive solution (since $u_\epsilon^m - u_\epsilon^1) > 0$) ϕ to the Jacobi equation $L_{\Sigma_\infty}\phi = 0$, which contradicts the bumpy metric assumption. If m is even then the top and bottom graphs have the same orientation, and instead one finds the equation their difference solves is the following:

$$L_{\Sigma_\infty}(u_\epsilon^m - u_\epsilon^1) + o(u_\epsilon^m - u_\epsilon^1) = -\epsilon h \cdot (h(x, u_\epsilon^m) + h(x, u_\epsilon^1)) \tag{5.5}$$

Applying a similar rescaling proceedure this time one finds a solution to the equation $L_{\Sigma_\infty}\phi = 2h$ such that ϕ does not change sign.

Now, the key point to dealing with this case (see lemma 4.2 in his paper) is that there is an h such that ϕ actually must change sign. To see this, by Sharp's compactness theorem and that the metric is bumpy there are only finitely many minimal surfaces Σ_i for a given area and index bound – these are the candidates for the minmax limit of course. On each Σ_i, we can find two disjoint open subsets $U_{i,1}, U_{i,2}$ so that the collection (varying over i as well) is pairwise disjoint. The idea then is to consider positive functions f_i^+ and negative functions f_i^- on $U_{i,1}, U_{i,2}$ respectively; defining $h_i^\pm = L_{\Sigma_i} f_i^\pm$ we see that any function ϕ solving $L_{\Sigma_i}\phi = h_i^\pm$ on these sets must be equal to $f_i^\pm$ by the definition of $h_i^\pm$ and that, by the bumpiness assumption, L_{Σ_i} is nondegenerate. By extending h appropriately, using that $\mathcal{S}(g)$ is dense, and potentially flipping the sign of h so that $\int h \geq 0$ (so we can run the PMC minmax argument) we produce a function h for which ϕ must change sign as needed. $\square$