SELF-SIMILAR SOLUTIONS OF MEAN CURVATURE FLOW

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ABSTRACT. We consider solutions of the mean curvature flow whose shape is preserved under the flow. These self-similar solutions are important as they provide models of both singularity formation and of singularity resolution for the flow. We discuss examples of these solutions and their basic properties as well as survey recent progress on classification results for self-shrinking flows.

1. INTRODUCTION

A *mean curvature flow* (MCF) is a smooth family of hypersurfaces $\{\Sigma_t\}_{t\in I}$ in $\mathbb{R}^{n+1}$ that satisfies the equation

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)^{\perp} = \mathbf{H}_{\Sigma_t}. \tag{1.1}$$

Here, $\left(\frac{\partial \mathbf{x}}{\partial t}\right)^{\perp}$ is the normal component of the velocity vector $\frac{\partial \mathbf{x}}{\partial t}$ of a point on the flow and $\mathbf{H}_{\Sigma_t}$ is the *mean curvature vector* of Σ_t. Recall, the mean curvature vector is the vector normal to the surface given by

$$\mathbf{H}_{\Sigma_t} = -H_{\Sigma_t}\mathbf{n}_{\Sigma_t} = \Delta_{\Sigma_t}\mathbf{x},$$

where $\mathbf{n}_{\Sigma_t}$ is a choice of unit normal, H_{Σ_t}, the *scalar mean curvature*, is the sum of the principle curvatures and Δ_{Σ_t} is the Laplace-Beltrami operator of the hypersurface Σ_t. We restrict attention to hypersurfaces in $\mathbb{R}^{n+1}$, that is to properly embedded codimension one submanifolds of $\mathbb{R}^{n+1}$ as this is the setting where the theory is most developed. This is a reasonable assumption because the parabolic maximum principle ensures that the flow of a hypersurface remains a hypersurface, though in general singularities will develop after a finite amount of time.

Mean curvature flow is the negative gradient flow of the area functional and has been extensively studied as a fundamental geometric heat equation. In this expository note, we will focus on self-similar solutions of mean curvature flow. These are solutions of (1.1) for which the shape of Σ_t is preserved. This is a special class of solutions that solve elliptic equations of minimal surface type. On the one hand, self-similar solutions are much easier to study, e.g., there are many interesting explicit solutions. On the other hand, the space of self-similar solutions is large enough to capture most of the interesting phenomena one encounters when studying (1.1).

Before preceding further, it is worth spending a little time understanding what (1.1) means. Analytically, a solution of (1.1) consists of a fixed smooth manifold, M, and a smooth map

$$\mathbf{F} : M \times [0, T) \to \mathbb{R}^{n+1} \tag{1.2}$$

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so that for each $t \in [0,T)$, $\mathbf{F}(\cdot,t) : M \to \Sigma_t$ is a parameterization of the hypersurface Σ_t and

$$(\partial_t \mathbf{F}(p,t))^\perp = \mathbf{H}_{\Sigma_t}(\mathbf{F}(p,t)) = \Delta_{\Sigma_t}\mathbf{F}(p,t). \tag{1.3}$$

This equation is not strictly parabolic. Indeed, the space of solutions is closed under pre-composing with a diffeomorphism of M. One often considers the more canonical equation where the velocity of points are normal to the family of hypersurfaces,

$$\partial_t \mathbf{F}(p,t) = \mathbf{H}_{\Sigma_t}(\mathbf{F}(p,t)) = \Delta_{\Sigma_t}\mathbf{F}(p,t).$$

Clearly, a solution of this equation is always a solution of (1.3). In fact, for reasonable Σ_0, e.g., compact and without boundary, a solution (1.3) can be turned into a solution of this equation by pre-composing with an appropriate time-varying family of diffeomorphisms.

More geometrically, one can consider the *space-time track* of a family of hypersurfaces $\{\Sigma_t\}_{t\in I}$ to be the hypersurface with boundary in *space-time* $\mathbb{R}^{n+1}_{\mathbf{x}} \times \mathbb{R}_t$ given by

$$\mathcal{S} = \{(\mathbf{x},t) : \mathbf{x} \in \Sigma_t, t \in I\} \subset \mathbb{R}^{n+1}_{\mathbf{x}} \times \mathbb{R}_t.$$

The family $\{\Sigma_t\}_{t\in I}$ is a mean curvature flow if and only if the vector field

$$T = \mathbf{H}_{\Sigma_t} + \frac{\partial}{\partial t}$$

along $\mathcal{S}$ is everywhere tangent to $\mathcal{S}$. Here t is the time coordinate and $\frac{\partial}{\partial t}$ is the corresponding coordinate vectorfield on the space-time.

2. Self-Similar Mean Curvature Flows

A natural first step in studying (1.1) is to study its symmetries. To that end consider the following natural actions on subsets of space, $\mathbb{R}^{n+1}$:

1. Spatial translation: For each $\mathbf{v} \in \mathbb{R}^{n+1}$ and $S \subset \mathbb{R}^{n+1}$ let $S+\mathbf{v} = \{\mathbf{x}+\mathbf{v} : \mathbf{x} \in S\}$;
2. Spatial dilation: For each $\rho > 0$ and $S \subset \mathbb{R}^{n+1}$ let $\rho S = \{\rho\mathbf{x} : \mathbf{x} \in S\}$.
3. Orthogonal transformation: For each $R \in O(n+1)$ and $S \subset \mathbb{R}^{n+1}$ let $R \cdot S = \{R \cdot \mathbf{x} : \mathbf{x} \in S\}$.

Likewise we consider corresponding natural actions on subsets of space-time $\mathbb{R}^{n+1} \times \mathbb{R}$:

1. Space-time translation: For each $(\mathbf{v},\tau) \in \mathbb{R}^{n+1} \times \mathbb{R}$ and $\mathcal{S} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ let $\mathcal{S} + (\mathbf{v},\tau) = \{(\mathbf{x}+\mathbf{v}, t+\tau) : (\mathbf{x},t) \in \mathcal{S}\}$
2. Parabolic dilation: For each $\rho > 0$ and $\mathcal{S} \subset \mathbb{R}^{n+1}\times\mathbb{R}$ let $\rho\mathcal{S} = \left\{(\rho\mathbf{x}, \rho^2 t) : (\mathbf{x},t) \in \mathcal{S}\right\}$;
3. Space-time orthogonal transformation: For each $R \in O(n+1)$ and $\mathcal{S} \subset \mathbb{R}^{n+1}\times\mathbb{R}$ let $R \cdot \mathcal{S} = \{(R\cdot\mathbf{x}, t) : (\mathbf{x},t) \in \mathcal{S}\}$.

MCF is invariant under several of the above symmetries:

Proposition 2.1. *If $\{\Sigma_t\}_{t\in[T_0,T_1)}$ is a MCF, then so is*

(1) $\{\Sigma_{t-\tau} + \mathbf{v}\}_{t\in[T_0+\tau,T_1+\tau)}$ for some fixed $\tau \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^{n+1}$;
(2) $\left\{\rho\Sigma_{\rho^{-2}t}\right\}_{t\in[\rho^2 T_0,\rho^2 T_1)}$ for fixed $\rho > 0$;
(3) $\{R\cdot\Sigma_t\}_{t\in[T_0,T_1)}$ for some fixed $R \in O(n+1)$.

That is, if $\mathcal{S}$ is the space-time track of a mean curvature flow, then so are its space-time translations, parabolic dilations and space-time orthogonal transformations.

Proof. We prove (2) as the others follow in similar fashion. Let $\mathbf{F} : M \times [T_0,T_1) \to \mathbb{R}^{n+1}$ be a parameterization of the flow $\{\Sigma_t\}_{t\in[T_0,T_1)}$. As such, $\mathbf{F}$ satisfies (1.2). Set $\mathbf{F}_\rho(p,t) =$

$\rho\mathbf{F}(p,\rho^{-2}t)$ so $\mathbf{F}_\rho : M \times [\rho^2 T_0, \rho^2 T_1) \to \mathbb{R}^{n+1}$ and $\mathbf{F}_\rho(M,t) = \rho\Sigma_{\rho^{-2}t}$. By the chain rule,

$$\left(\frac{\partial}{\partial t}\mathbf{F}_\rho(p,t)\right)^\perp = \rho^{-1}\left(\frac{\partial}{\partial t}\mathbf{F}(p,\rho^{-2}t)\right)^\perp = \rho^{-1}\mathbf{H}_{\Sigma_{\rho^{-2}t}}(\mathbf{F}(p,\rho^{-2}t))$$
$$= \mathbf{H}_{\rho\Sigma_{\rho^{-2}t}}(\rho\mathbf{F}(p,\rho^{-2}t)) = \mathbf{H}_{\rho\Sigma_{\rho^{-2}t}}(\mathbf{F}_\rho(p,\rho^{-2}t)).$$

That is, $\mathbf{F}_\rho$ satisfies (1.1). □

In order to find solutions of MCF it is natural to first study those that evolve while maintaining their shape – called *self-similar solutions* or *solitons*. This turns out to not only produce a large number of interesting solutions, but to also give deep insight into the regularity properties of MCF and into its long time behavior.

Most generally, consider $\Sigma \subset \mathbb{R}^{n+1}$ so that

$$\{\Sigma_t\}_{t\in I} = \{\rho(t)(R(t)\cdot\Sigma) + \mathbf{v}(t)\}_{t\in I} \text{ is a MCF.}$$

Here, $\rho : I \to \mathbb{R}^+$, $R : I \to O(n+1)$ and $\mathbf{v} : I \to \mathbb{R}^{n+1}$ are to be determined. Plugging this ansatz into (1.2), yields differential equations for the unknown functions. Halldorsson [32,33] gives a complete treatment of this. We restrict attention to three special cases of self-similar flows as these have proven to be the most important.

Proposition 2.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a properly embedded hypersurface.*

1. *The flow $\left\{\sqrt{-t}\Sigma\right\}_{t<0}$ is a MCF if and only if $\mathbf{H}_\Sigma = -\frac{\mathbf{x}^\perp}{2}$. In this case, Σ is called a* self-shrinker.
2. *The flow $\left\{\sqrt{t}\Sigma\right\}_{t>0}$ is a MCF if and only if $\mathbf{H}_\Sigma = \frac{\mathbf{x}^\perp}{2}$. In this case the flow is called a* self-expander.
3. *The flow $\{\Sigma + t\mathbf{v}\}_{t\in\mathbb{R}}$ is a MCF if and only if $\mathbf{H}_\Sigma = \mathbf{v}^\perp$. In this case the flow is called a* translator *and the vector* $\mathbf{v}$ *is its* velocity.

Remark 2.3. When $\mathbf{v} = \mathbf{0}$ a translator is a minimal hypersurface.

Proof. We prove something slightly stronger, namely any flow that moves by dilation is, modulo parabolic dilation and time translation, the flow self-shrinker or self-expander and that any flow that moves by translation is a translator.

Indeed, suppose

$$\{\lambda(t)\Sigma\}_{t\in I}$$

is a MCF. Parameterize Σ by $\mathbf{F} : M \to \mathbb{R}^{n+1}$ so $\lambda(t)\mathbf{F} : M \times I \to \mathbb{R}^{n+1}$ parameterizes the flow. This satisfies (1.2) if and only if

$$\lambda'(t)\mathbf{F}^\perp(p) = (\partial_t(\lambda(t)\mathbf{F}(p)))^\perp$$
$$= \mathbf{H}_{\lambda(t)\Sigma_t}(\lambda(t)\mathbf{F}(p)) = \frac{1}{\lambda(t)}\mathbf{H}_\Sigma(\mathbf{F}(p))$$

In particular, one must have $\lambda(t) > 0$ and $\lambda(t)\lambda(t)' = C$ be independent of time for this to be a MCF. All solutions of this ODE that satisfy the given constraint are of the form

$$\lambda(t) = \sqrt{C(t-t_0)}$$

for $C \neq 0$ where the solution is defined for $t > t_0$ when $C > 0$ and the solution is defined for $t < t_0$ when $C > 0$. By parabolically dilating by a factor of $|C|^{1/2}$ and time translating by t_0 one sees that the flow is of the form

$$\sqrt{\pm t}\Sigma_{t\in I'}$$

and, moreover, the surface Σ, after dilating appropriately satisfies either the self-shrinker equation (when $C < 0$) or the self-expander equation (when $C > 0$). Translators are treated in a similar fashion. $\square$

Non-flat self-shrinkers have associated MCFs that become singular at $t = 0$. As such, they provide examples of how singularities form. In fact, by combining Huisken's monotonicity formula [39] with suitable weak notions of the flow, they provide all singularity models of the flow. The space-time track of a self-shrinker is invariant under parabolic dilation and so is, in many ways, plays a role analogous to that of minimal cones in the regularity theory of minimal surfaces. Self-shrinkers are also prototypical examples of *ancient solutions*. That is, flows that extend indefinitely into the past.

In contrast, non-flat self-expanders "flow out" of a singularity at $t = 0$. As such, they provide models of how conical singularities of MCF resolve. They are expected to describe either how an initially mildly singular hypersurface is smoothed out or how the flow might resolve a conical singularity that has formed. However, at present an adequate monotonicity formula has not been formulated and so there are few results in this direction. Self-expanders are also prototypical examples of *immortal solutions*, that is flows that extend indefinitely into the future.

Finally, translators arise naturally in studying how "cylinderical" singularities resolve. They are particularly important in carrying out surgery procedures for the flow, e.g., [13, 34, 35, 41]. Translators also arise in the natural elliptic regularization scheme for MCF [15,28–31,42]. They are prototypical examples of *eternal solutions*, that is flows that exist forever both backwards and forwards in time.

2.1. **Examples of self-shrinkers.** We describe some key examples of self-shrinking solutions. The self-shrinker equation is invariant under orthogonal transformations so each example naturally lives in $O(n+1)$ invariant family.

The most basic examples are given by the family of generalized cylinders that degenerates to the flat plane and to the round sphere.

Example 2.4 (Generalized Cylinders)**.** For $0 \leq k \leq n$ the hypersurfaces

$$\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k} = \left\{ x_1^2 + \ldots + x_{k+1}^2 = 2k \right\}$$

are all self-shrinkers. When $k = 0$, the generalized cylinder is the flat plane

$$\mathbb{R}^n = \{ x_{n+1} = 0 \}$$

and when $k = n$ the generalized cylinder is the round sphere of radius $\sqrt{2n}$

$$\sqrt{2n}\mathbb{S}^n = \left\{ x_1^2 + \ldots + x_{n+1}^2 = 2n \right\}.$$

This family forms the simplest examples of self-shrinkers. They already illustrate the three qualitative features of a self-shrinker near infinity: either it is *asymptotically conical* as in $\mathbb{R}^n$, it is *closed* (i.e., compact and without boundary) as with $\mathbb{S}^n$ or it is *asymptotically cylindrical* as in $\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$ for $1 \leq k \leq n-1$.

Symmetry methods produces several topologically non-trivial closed self-shrinkers.

Example 2.5 (Angenent Torus [2])**.** In $\mathbb{R}^{n+1}$ there is at least one rotationally invariant closed self-shrinker. $\mathbb{T}^n_A$ that is topologically, $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.

This shrinker is found by solving (non-explicitly) an ODE. The nature of the construction does not preclude the possibility that there is more than one embedded rotationally symmetric embedded self-shrinking torus (in fact there are many immersed self-shrinking tori [25]), though uniqueness is expected. We abuse notation slightly and refer to any such torus by $\mathbb{T}^n_A$. A similar technique was employed by McGrath [47] to produce a closed self-shrinker in $\mathbb{R}^{2n}$ that is topological $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{S}^1$.

Using desingularization methods, an infinite family of asymptotically conical self-shrinkers of high genus have been constructed in $\mathbb{R}^3$:

Example 2.6 (Kapouleas-Kleene-Møller [44])**.** In $\mathbb{R}^3$ there is a sequence of self-shrinkers,

$$\Sigma_g^{KKM}\subset\mathbb{R}^3$$

of any genus $g\gg 1$ with one conical end. This family is obtained by desingularizing $\mathbb{R}^2\cup 2\mathbb{S}^2$ in the sense $\lim_{g\to\infty}\Sigma^{KKM}=\mathbb{R}^2\cup 2\mathbb{S}^2$.

Using min-max methods, Ketover [45] has been able to produce self-shrinkers of positive, low, genus with the symmetries of the platonic solids. However, it is not known whether this construction produces closed examples.

Example 2.7 (Ketover [45])**.** There exist self-shrinkers

$$\Sigma_g^K\subset\mathbb{R}^3$$

of genus $g=3,5,7,11$ and 19.

Finally we remark that there are many numerical examples of self-shrinkers – e.g., [3, 16, 43]. While some of the examples constructed above, e.g. Σ^K and Σ^{KKM} seem to realize these numerical pictures, there are many numerical examples that have yet to be given a rigorous construction. A particularly interesting numerical example that has not yet been made rigorous is given by Angenent-Chopp-Ilmanen [3] who exhibit a genus-one self-shrinker that flows into a cone out of which many different self-expanders can emerge.

2.2. **Examples of self-expanders.** In contrast to self-shrinkers, there are no closed self-expanders, but there are many asymptotically conical self-expanders.

Example 2.8 (Asymptotic Plateau Problem [22, 43])**.** For $2\le n\le 6$, every regular cone $C\subset\mathbb{R}^{n+1}$ admits at least one self-expander, Σ, which is asymptotic to C in that

$$\lim_{\rho\to 0}\rho\Sigma=C \iff \lim_{t\to 0^+}\sqrt{t}\Sigma=C.$$

In particular, the MCF associated to Σ flows out of the cone C. This is proved by solving an asymptotic plateau problem – the idea for which was sketched by Ilmanen by [43] and solved by Ding [22]. In particular, the self-expander constructed is *stable*.

There are cones C for which there are more than one self-expander asymptotic to C.

Example 2.9 (Angenent-Chopp-Ilmanen [3])**.** Let

$$D_\alpha=\left\{x_{n+1}^2\sin^2\alpha=(x_1^2+\ldots+x_n^2)\cos^2\alpha\right\}$$

be the rotationally symmetric double cone making angle $\alpha\in(0,\pi/2)$ with x_{n+1}-axis. For each α there is a disconnected rotationally symmetric self-expannder asymptotic to D_α. There is a critical angle $\alpha_{crit}\approx 66^\circ$ so for $\alpha_{crit}<\alpha<\frac{\pi}{2}$ there is also a connected rotationally symmetric expander asymptotic to D_α.

In [36], Helmensdorfer constructs a second connected rotationally symmetric expander $\alpha_{crit}<\alpha<\frac{\pi}{2}$ asymptotic to D_α. In particular one does not have uniqueness even in a fixed topological class. In [11], L.Wang and I show that this is generic, that is we give an open set of cones in $\mathbb{R}^3$ so that there are at least three self-expanders, two connected annuli and one consisting of a pair of disconnected disks asymptotic to each cone in the open set.

2.3. **Examples of translators.** We conclude by giving examples of translators. By rescaling and rotating appropriately, we may assume that the translating velocity, $\mathbf{v}$, is either $\mathbf{0}$ or $\mathbf{e}_1$. In the former case, the translator is a minimal hypersurface – a well studied object – and so we focus on the latter.

While minimal surfaces have velocity $\mathbf{0}$, cylinders over a minimal hypersurface can have non-zero velocity.

Example 2.10 (Minimal surface cylinders)**.** If $\Sigma \subset \mathbb{R}^{n+1}$ satisfies

$$\Sigma = \Sigma' \times \mathbb{R} = \left\{(\mathbf{x}, x_{n_{n+1}}) : \mathbf{x} \in \Sigma' \subset \mathbb{R}^n\right\}$$

where $\Sigma' \subset \mathbb{R}^n$ is minimal, then Σ' is a translator with velocity $\mathbf{v} = \mathbf{e}_{n+1}$. In this case, the flow is static even though the velocity is not.

There exist two families of explicit solutions:

Example 2.11 (Grim Reaper and Grim Reaper Cylinder)**.** The curve

$$\Sigma^{GR} = \left\{x_1 = \log\sec x_2 : |x_2| < \frac{\pi}{2}\right\} \subset \mathbb{R}^2.$$

is a translator with velocity $\mathbf{v} = \mathbf{e}_1$ called the grim reaper. It generalizes to the grim reaper cylinder

$$\Sigma^{GR} \times \mathbb{R}^{n-1} = \left\{(x_1, \ldots, x_{n+1}) : (x_1, x_2) \in \Sigma^{GR}\right\} \subset \mathbb{R}^{n+1}$$

that is a translater in $\mathbb{R}^{n+1}$ with velocity $\mathbf{v} = \mathbf{e}_1$. The grim reaper cylinder is a (weakly) convex graph over the slab $\left\{|x_2| \leq \frac{\pi}{2}\right\} \subset \{x_1 = 0\}$.

Example 2.12 (Generalized Grim Reaper Cylinder [53])**.** For $n \geq 2$ and $\lambda \geq 1$ the family of hypersurfaces

$$\Sigma^{GGR}_\lambda = \left\{x_1 = \log\sec \frac{x_2}{\lambda} + \sqrt{\lambda^2 - 1}x_3 : |x_2| < \lambda\frac{\pi}{2}\right\}$$

are translators with velocity $\mathbf{v} = \mathbf{e}_1$. Each surface is a (weakly) convex graph over the slab $\left\{|x_2| \leq \lambda\frac{\pi}{2}\right\} \subset \{x_1 = 0\}$ and is obtained by rotating and scaling $\Sigma^{GR} \times \mathbb{R}^{n-1}$.

Using symmetry methods one produces several interesting families of examples:

Example 2.13 (Bowl Translator)**.** There is a translator, Σ^B, called the bowl soliton, that has velocity $\mathbf{v} = \mathbf{e}_1$ and is an entire convex graph over the plane $\{x_1 = 0\}$. It is rotationally symmetric about the $\mathbf{e}_1$-axis and is asymptotic to a parabola at infinity.

Example 2.14 (Translating Catenoid [17])**.** There is a one parameter family of rotationally symmetric expanders, Σ^C_δ with velocity $\mathbf{v} = \mathbf{e}_1$. These solutions look like two bowl translators joined by a neck of size δ. In particular, $\lim_{\delta\to 0} \Sigma^C_\delta$ converges to a multiplicity two copy of Σ^B.

In fact, Σ^B and $\Sigma^{GR} \times \mathbb{R}^{n-1}$ sit in a one parameter family:

Example 2.15 (Δ-Wing [14, 38])**.** There is a family, Σ^{DW}_λ, for $1 < \lambda < \infty$ called the delta wings that are translators with velocity $\mathbf{v} = \mathbf{e}_1$. For each λ, Σ^{DW}_λ is a convex graph over the slab $\left\{|x_2| \leq \lambda\frac{\pi}{2}\right\} \subset \{x_1 = 0\}$. This family interpolates between the grim reaper cylinder and the bowl soliton.

Desingularization methods have been very successful in producing translators in $\mathbb{R}^3$:

Example 2.16 (Translating Tridents [49])**.** There is a discrete family $\Sigma^{TR}_{\tau_i}$ of translators in $\mathbb{R}^3$ with infinite genus called tridents. Here τ_i is a sequence tending to 0. These translators have velocity $\mathbf{v} = \mathbf{e}_1$ and are invariant under the discrete symmetry $\mathbf{x} \mapsto \mathbf{x} + \tau_i \mathbf{e}_3$. As $x_1 \to \infty$ they are asymptotic to three planes parallel to $\{x_2 = 0\}$ and to one plane parallel to $\{x_2 = 0\}$ as $x_1 \to -\infty$. They are constructed by desingularizing the union of a grim reaper cylinder and a plane and so $\lim_{\tau_i \to 0} \Sigma^{TR}_{\tau} = (\Sigma^{GR} \times \mathbb{R}) \cup \{x_2 = 0\}$.

Nguyen [50, 51] has generalized this construction to produce many disingularizations of unions of copies of grim reaper cylinders in $\mathbb{R}^3$. Davila-Del Pino-Nguyen [21] and G. Smith [52] have independently produced families of translators with finite genus by desingularizing $\Sigma^B \cup \Sigma^C_\delta$ in an appropriate manner.

3. Variational and stability properties of self-similar solutions

Self-shrinkers, self-expanders and translators are critical points of natural geometric variational problems. This observation allows one to make use of techniques developed for the geometric calculus of variations and especially for minimal hypersurfaces. For instance, several of the constructions in the previous section use this point of view. There are also links between the stability properties as variational problems and dynamical stability properties of the associated flows.

We begin by considering the following functional on hypersurfaces $\Sigma \in \mathbb{R}^{n+1}$:

$$F[\Sigma] = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|\mathbf{x}|^2}{4}} d\mathrm{vol}_\Sigma,$$

sometimes called the *Gaussian surface area of* Σ. Observe that $F[\mathbb{R}^n] = 1$ so this functional can be finite on non-compact Σ. If $\mathbf{X} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a compactly supported vector field with flow ϕ_t, then the first variation formula gives

$$\frac{d}{dt}|_{t=0} F[\phi_t(\Sigma)] = (4\pi)^{-\frac{n}{2}} \int_\Sigma \left(\mathbf{H}_\Sigma + \frac{\mathbf{x}^\perp}{2} \right) \cdot \mathbf{X} e^{-\frac{|\mathbf{x}|^2}{4}} d\mathrm{vol}_\Sigma.$$

That is, the critical points of F are precisely the self-shrinkers. Similarly, let

$$E[\Sigma] = \int_\Sigma e^{\frac{|\mathbf{x}|^2}{4}} d\mathrm{vol}_\Sigma \text{ and } T_{\mathbf{v}}[\Sigma] = \int_\Sigma e^{\mathbf{x}\cdot\mathbf{v}} d\mathrm{vol}_\Sigma.$$

Computing as above yields that the critical points of $T_{\mathbf{v}}$ are the translators with velocity $\mathbf{v}$ while the critical points of E are the expanders. In contrast with self-shrinkers, there are no closed self-expanders or self-translaters. Furthermore, the functionals E and $T_{\mathbf{v}}$ will always take value ∞ on their critical points.

It is also possible to consider the following (possibly incomplete) metrics on $\mathbb{R}^{n+1}$ that are conformal to the euclidean metric g_{Euc}:

$$g_S = (4\pi)^{-1/2} e^{-\frac{|\mathbf{x}|^2}{4n}} g_{Euc}, \;\; g_E = e^{\frac{|\mathbf{x}|^2}{4n}} g_{Euc} \text{ and } g_T^{\mathbf{v}} = e^{\frac{\mathbf{x}\cdot\mathbf{v}}{n}} g_{Euc}.$$

It's not hard to verify that self-shrinkers are precisely the minimal surface of g_S, self-expanders are the minimal surfaces for g_E and translators are minimal surfaces for $g_T^{\mathbf{v}}$.

The metric g_S is "mostly" positively curved (in its bulk), suggesting similarities between self-shrinkers and minimal surfaces in $\mathbb{S}^{n+1}$. Likewise, g_E is "mostly" negatively curved, linking self-expanders with minimal surfaces in $\mathbb{H}^{n+1}$. Minimal hypersurfaces in the sphere are much more "rigid" than those in hyperbolic space. A similar phenomena occurs for self-shrinkers and self-expanders. There is a parabolic interpretation of this phenomena. Namely, forward in time evolution of a heat flow (i.e., flows corresponding to expanders) is a well-posed problem while backwards in time evolution of a heat flow (i.e., flows corresponding to shrinkers) is ill-posed.

Before turning to the rigidity properties of self-shrinkers which occupies the remainder of the article, I briefly illustrate some of the flexible nature of self-expanders. Recall, Example 2.8 shows that, for each regular cone C, there is a self-expander, Γ, asymptotic to C. This contrasts with the case of self-shrinkers where the asymptotic cones are severely restricted (e.g., are real analytic). In addition, as this Γ is produced by a minimization procedure it is stable and so there are many stable self-expanders. This again contrasts with self-shrinkers where, as we will see below, there are no stable self-shrinkers. In fact, more is true. L.Wang and I [9] have shown that the space of all asymptotically conical expanders (suitably understood) has the structure of an infinite dimensional Banach manifold. In particular, for a "generic" cone C_0 in $\mathbb{R}^{n+1}$ and expander Γ_0 asymptotic to C_0, for any smooth perturbation C_t of the cone C_0, there is a corresponding smooth family Γ_t of Γ_0 consisting of expanders asymptotic to C_t.

3.1. **Self-shrinkers of low index.** We now study stability properties of self-shrinkers from a variational point of view. In particular, we give a complete classification of self-shrinkers of low index which follows [20]. Note that this is an example of the rigidity properties of self-shrinkers, as Example 2.8 shows that there are many stable self-expanders and such a classification for self-expanders is hopeless.

Fix $\Sigma \subset \mathbb{R}^{n+1}$ a self-shrinker and suppose $\mathbf{n}_\Sigma$ is the unit normal. For each $u \in C^\infty_0(\Sigma)$ consider the normal variation:

$$\Sigma_t = \{\mathbf{x}(p) + tu(p)\mathbf{n}_\Sigma(p) : p \in \Sigma\}.$$

One computes

$$\frac{d^2}{dt^2}|_{t=0}F[\Sigma_t] = Q_\Sigma[u] = (4\pi)^{-n/2}\int_\Sigma \left(|\nabla_\Sigma u|^2 - |A_\Sigma|^2u^2 - \frac{1}{2}u^2\right)e^{-\frac{|\mathbf{x}|^2}{4}}d\mathrm{vol}_\Sigma.$$

When Σ is closed, $Q_\Sigma[1] < 0$ and so all closed self-shrinkres are unstable. In fact, when Σ non-compact but $F[\Sigma] < \infty$, then one can show that Σ has polynomial volume growth and so as the weight decays rapidly $Q_\Sigma[\phi] < 0$ for an appropriate cutoff ϕ. That is, all reasonable self-shrinkers are unstable.

Integrating by parts yields

$$Q_\Sigma[u] = \int_\Sigma u(-L_\Sigma u)e^{-\frac{|\mathbf{x}|^2}{4}}d\mathrm{vol}_\Sigma$$

where L_Σ is the *Jacobi operator* given by

$$L_\Sigma u = \Delta_\Sigma u - \frac{\mathbf{x}^\top}{2}\cdot\nabla_\Sigma u + |A_\Sigma|^2 + \frac{1}{2}.$$

As usual, the variational properties of Σ are related to the spectral properties of L_Σ in the right space. This is straightforward when Σ is compact but is still true if $F[\Sigma] < \infty$ due to decay of the weight. In particular, when $F[\Sigma] < \infty$, L_Σ has discrete spectrum in an appropriately weighted L^2 space even if Σ is non-compact.

Define the *Morse index* of Σ to be

$$\mathrm{Ind}(\Sigma) = \max\{\dim V : V \subset C^\infty_0(\Sigma), Q_\Sigma[u] < 0, \forall 0 \neq u \in V\}.$$

Using the variational characterization of eigenvalues and the spectral properties of L_Σ,

$$\mathrm{Ind}(\Sigma) = \dim E^-(-L_\Sigma)$$

where $E^-(-L_\Sigma)$ is the space of eigenfunctions of $-L_\Sigma$ with negative eigenvalues. On self-shrinkers certain geometrically defined functions always satisfy the eigenfunction equation:

Proposition 3.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be self-shrinker. If $H_\Sigma = -\mathbf{H}_\Sigma \cdot \mathbf{n}_\Sigma = \frac{\mathbf{x}\cdot\mathbf{n}_\Sigma}{2}$ and $T_i = \mathbf{n}_\Sigma \cdot \mathbf{e}_i$, $i = 1, \ldots, n+1$, then*

$$-L_\Sigma H_\Sigma = -H_\Sigma \text{ and } -L_\Sigma T_i = -\frac{1}{2} T_i.$$

If in addition, $\Sigma = \Sigma' \times \mathbb{R}^{n-k}$ for $\Sigma' \subset \mathbb{R}^{k+1}$, then, for $k+2 \leq j \leq n+1$, the functions $R_j = H_\Sigma x_j$ satisfy $-L_\Sigma R_j = -\frac{1}{2} R_j$.

Proof. The first claim follows from straightforward computations involving Simons' identity – see [20, Lemma 5.5]. The second claim follows from the fact that, for a self-shrinker,

$$\Delta_\Sigma x_j - \frac{\mathbf{x}}{2} \cdot \nabla_\Sigma x_j = -\frac{1}{2} x_j$$

for all $1 \leq j \leq n+1$ and from the observation that, when Σ splits,

$$\nabla_\Sigma H_\Sigma \cdot \nabla_\Sigma x_j = 0,$$

for all $k+2 \leq j \leq n+1$. Indeed,

$$\begin{aligned} -L_\Sigma R_j &= -x_j L_\Sigma H_\Sigma - 2\nabla_\Sigma H_\Sigma \cdot \nabla_\Sigma x_j - H_j \left(\Delta_\Sigma - \frac{\mathbf{x}}{2} \cdot \nabla_\Sigma \right) x_j \\ &= -x_j H_\Sigma + \frac{1}{2} x_j H_\Sigma = -\frac{1}{2} R_j. \end{aligned}$$

□

That is, if H_Σ doesn't identically vanish and Σ is regular enough at infinity so that H_Σ lies in the right space, then it is an eigenfunction of $-L_\Sigma$ with eigenvalue -1. Similarly, there are, at most $n+1$, eigenfunctions $T_1, \ldots, T_{n+1}$ with eigenvalue $-\frac{1}{2}$.

With this in mind, let $G = \mathrm{span}(H_\Sigma, T_1, \ldots, T_n)$, be the space of *geometric eigenfunctions*. Notice, $\dim G \leq n+2$ with equality on $\sqrt{2n}\mathbb{S}^n$ but strict inequality on $\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$ for $0 \leq k < n$. In fact, on $\Sigma = \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$, L_Σ has constant coefficients and so one can explicitly work out that all the eigenfunctions with negative eigenvalues are linear combinations of those given by Proposition 3.1. As such,

(1) $\mathrm{Ind}(\mathbb{R}^n) = 1$;
(2) For $1 \leq k \leq n$, $\mathrm{Ind}(\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}) = n+2$.

In fact, this is the complete list of self-shrinkers of low Morse index – this was shown, though not stated in this form, by Colding-Minicozzi in [20].

Theorem 3.2. *If $\Sigma \subset \mathbb{R}^{n+1}$ is a self-shrinker with $\mathrm{Ind}(\Sigma) \leq n+2$, then, up to a rotation,*

$$\Sigma = \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$$

where here $0 \leq k \leq n$ (i.e., we include $\mathbb{R}^n$ and $\sqrt{2n}\mathbb{S}^n$).

Proof. If $\dim G = n+2$, then the index bound implies H_Σ is lowest eigenfunction and so up to switching the sign of the unit normal, $H_\Sigma > 0$. When Σ is closed this is an immediate consequence of standard spectral theory. When Σ is non-compact one must use the fact that the natural weight in the problem decays very rapidly at infinity. As a consequence, Σ is mean convex and so after rotating, $\Sigma = \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$ for $1 \leq k \leq n$ by the classification of mean convex self-shrinkers [20,39] – see Theorem 4.5 below. However, if $k < n$, then $T_{k+2}, \ldots, T_{n+1}$ must all vanish and so $\Sigma = \sqrt{2n}\mathbb{S}^n$.

If Σ is compact, then, by what we have already shown, $H_\Sigma \neq 0$ for at least one point, e.g., at the point at maximal distance from $\mathbf{0}$. Similarly, all the $T_1, \ldots, T_{n+1}$ must be linearly independent. Indeed, if $c_1T_1+\cdots+c_{n+1}T_{n+1} = 0$, then $\mathbf{v} = c_1\mathbf{e}_1+\cdots c_{n+1}\mathbf{e}_{n+1}$ is tangent to Σ everywhere. This is only possible if $\mathbf{v} = \mathbf{0}$. Thus, for Σ compact, $\dim G = n+2$ and so by the previous paragraph Σ must be a sphere.

If Σ is non-compact, then $\dim G < n+2$. Hence, either $H_\Sigma = 0$ identically or H_Σ is not identically zero and there is a linear dependence among the $T_1, \ldots T_{n+1}$. In the former case, the self-shrinker equation implies Σ is a cone and hence, as Σ is smooth, it must be $\mathbb{R}^n$. In the later case, the argument of the previous paragraph implies that, after rotation, Σ splits as a product $\Sigma_1 \times \mathbb{R}$ with $\Sigma_1 \subset \mathbb{R}^n$. Repeating this argument implies that, after rotation, we may split $\Sigma = \Sigma' \times \mathbb{R}^{n-k}$ with Σ' a self-shrinker in $\mathbb{R}^{k+1}$ so that $\dim G(\Sigma') = k+2$.

In particular, there are $k+2$ linearly independent eigenfunctions $H_{\Sigma'}, T_1', \ldots, T_{k+1}'$ on Σ'. Clearly,

$$H_\Sigma(p, x_{k+2}, \ldots, x_{n+1}) = H_{\Sigma'}(p)$$

and, for $1 \leq j \leq k+1$,

$$T_j(p, x_{k+2}, \ldots, x_{n+1}) = T_j'(p).$$

By Proposition 3.1 and the fact that H_Σ does not identically vanish, for $k+2 \leq j \leq n+1$, $R_j = H_\Sigma x_j$ are eigenfunctions of $-L_\Sigma$ with eigenvalue $-\frac{1}{2}$ that do not vanish identically. In fact, the set of $n+2$ non-zero functions

$$\{H_\Sigma, T_1, \ldots, T_{n+1-\ell}, R_{n+2-\ell}, \ldots, R_{n+1}\}$$

can be shown to be orthogonal in the weighed L^2 space and so this set is linearly independent. As $\mathrm{Ind}(\Sigma) = n+2$, H_Σ is then the lowest eigenfunction of $-L_\Sigma$ and so Σ is mean convex and hence $\Sigma = \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$ for $1 \leq k \leq n-1$. In particular, $\Sigma' = \sqrt{2k}\mathbb{S}^k$. $\square$

The proof of the previous theorem implies that, for compact self-shrinkers and non-compact self-shrinkers that do not split, there are always at least $n+2$ directions of instability coming from translation and dilation. As such it makes sense to say a self-shrinker Σ is *F-stable* if those are the only instablities. That is, if, for all u orthogonal, in a weighted L^2-sense, to G, $Q_\Sigma[u] \geq 0$. Clearly, such a shrinker has $\mathrm{Ind}(\Sigma) \leq n+2$, but the converse need not be true as evidenced by the generalized cylinders $\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$ for $1 \leq k \leq n-1$.

Theorem 3.3 (Colding-Minicozzi [20]). *If Σ is F-stable, then $\Sigma = \mathbb{R}^n$ or $\Sigma = \sqrt{2n}\mathbb{S}^n$.*

3.2. **Stability of singularity models and entropy.** We say a self-shrinker is *dynamically stable* if MCF evolves a small (compactly supported) perturbation of the self-shrinker "back" to the original singularity model. For instance, $\sqrt{2n}\mathbb{S}^n$ is dynamically stable because any small perturbation of it is convex and so the flow of the perturbation disappears in a round point [40]. Note that the location of the singularity of the perturbed surface may move in space-time. Indeed, for closed self-shrinkers the singularity always moves if one perturbs by a space-time translation. In order to study the dynamical stability of more general singularity models, Colding-Minicozzi in [20] introduced a functional they called the *entropy* which, in contrast with F, accounts for these geometric instabilities.

Given a hypersurface $\Gamma \subset \mathbb{R}^{n+1}$, define its *entropy* to be

$$\lambda[\Gamma] = \sup_{\mathbf{y}\in\mathbb{R}^{n+1}, \rho>0} F[\rho\Gamma + \mathbf{y}] = \sup_{\mathbf{x}_0\in\mathbb{R}^{n+1}, \tau>0} \int_\Gamma \frac{e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4\tau}}}{(4\pi\tau)^{n/2}} dvol_\Gamma$$

where the second equality follows by a change of variables. This quantity may be thought of as a rough measure of geometric complexity and it is interesting to study its properties for general hypersurfaces – see Section 4.2 below.

It follows from Huisken's monotonicity formula that entropy is non-decreasing along the MCF and that if Σ is a self-shrinker, then $\lambda[\Sigma] = F[\Sigma]$.

Proposition 3.4. *If $\{\Gamma_t\}_{t\in[0,T)}$ is a MCF in $\mathbb{R}^{n+1}$, then $\lambda[\Gamma_t] \leq \lambda[\Gamma_{t'}]$ for $0 \leq t \leq t' < T$. In addition, if $\Sigma \subset \mathbb{R}^{n+1}$ is a self-shrinker, then $\lambda[\Sigma] = F[\Sigma]$.*

Proof. By Huisken's monotonicity formula [39], for $t_0 = t' + \tau > t'$ and $\mathbf{x}_0 \in \mathbb{R}^{n+1}$,

$$\begin{aligned}\int_{\Gamma_{t'}} \frac{e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4\tau}}}{(4\pi\tau)^{n/2}} dvol_{\Gamma_{t'}} &= \int_{\Gamma_{t'}} \frac{e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4(t_0-t')}}}{(4\pi(t_0-t'))^{n/2}} dvol_{\Gamma_{t'}} \\ &\leq \int_{\Gamma_t} \frac{e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4(t_0-t)}}}{(4\pi(t_0-t))^{n/2}} dvol_{\Gamma_t} \leq \lambda[\Gamma_t].\end{aligned}$$

where the last inequality used $t_0 - t > t_0 - t' = \tau > 0$. As $\tau > 0$ and $\mathbf{x}_0$ were arbitrary, taking the supremum gives

$$\lambda[\Gamma_t'] \leq \lambda[\Gamma_t],$$

proving the first claim.

For the second claim first observe that for any hypersurface, Σ, the definition of λ, implies $F[\Sigma] \leq \lambda[\Sigma]$. When Σ is a self-shrinker, $\left\{\Sigma_t = \sqrt{-t}\Sigma\right\}_{t<0}$ is a MCF and so for any $\mathbf{x}_0 \in \mathbb{R}^{n+1}$ and $\tau > 0$, Huisken's monotonicity formula implies

$$\begin{aligned}\int_\Sigma \frac{e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4\tau}}}{(4\pi\tau)^{n/2}} dvol_\Sigma &\leq \limsup_{t\to-\infty} \int_{\Sigma_t} \frac{e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4(\tau-t-1})}}{(4\pi(\tau-t-1)^{n/2}} dvol_{\Sigma_t} \\ = \limsup_{t\to-\infty} \int_\Sigma \frac{e^{-\frac{|\mathbf{x}-(-t)^{-1/2}\mathbf{x}_0|^2}{4(1+(-t)^{-1}(\tau-1)}}}{(4\pi(1+(-t)^{-1}(\tau-1))^{n/2}} dvol_\Sigma &= \int_\Sigma \frac{e^{-\frac{|\mathbf{x}|^2}{4}}}{(4\pi)^{n/2}} dvol_\Sigma.\end{aligned}$$

where the second equality follows form a change of variables and the fact that $\Sigma_t = \sqrt{-t}\Sigma$ and the third equality follows form the dominated convergence theorem. From this it immediately follows that $\lambda[\Sigma] \leq F[\Sigma]$ verifying the second claim. $\square$

As a consequence,

$$\lambda[\mathbb{S}^n] = F[\sqrt{2n}\mathbb{S}^n]$$

and so by computations of Stone [54],

$$2 > \lambda[\mathbb{S}^1] > \frac{3}{2} > \lambda[\mathbb{S}^2] > \cdots > \lambda[\mathbb{S}^n] > \lambda[\mathbb{S}^{n+1}] > \cdots \to \sqrt{2}.$$

Furthermore, for $1 \leq k \leq n$,

$$\lambda[\mathbb{S}^k \times \mathbb{R}^{n-k}] = F[\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}] = F[\sqrt{2k}\mathbb{S}^k] = \lambda[\mathbb{S}^k]$$

while $\lambda[\mathbb{R}^n] = 1$ and so

$$\lambda[\mathbb{R}^n] = 1 < \sqrt{2} < \lambda[\mathbb{S}^n] < \lambda[\mathbb{S}^{n-1} \times \mathbb{R}] < \cdots < \lambda[\mathbb{S}^1 \times \mathbb{R}^{n-1}].$$

That is, the round sphere has the lowest entropy among the non-flat generalized cylinders.

In [20], Colding-Minicozzi study self-shrinkers that are stable for λ. This is slightly subtle as λ is only Lipschitz so one cannot directly define second variation. They say a self-shrinker Σ, is *entropy stable* if either:

(1) Σ is not of the form $\Sigma' \times \mathbb{R}$ and for every compactly supported smooth vector field $\mathbf{X}$ with flow ϕ_t one has $\lambda[\phi_t(\Sigma)] \geq \lambda(\Sigma)$ for $|t|$ sufficiently small; or
(2) $\Sigma = \Sigma' \times \mathbb{R}^l$ and $\Sigma' \subset \mathbb{R}^{n-l+1}$ satisfies (1);

Observe that by translating along the translation invariant factor, for any self-shrinker of the form $\Sigma' \times \mathbb{R}$ it is not possible to decrease its entropy with a compactly supported variation. This is the reason for the condition that Σ not split off a line in (1).

As any perturbation of $\sqrt{2n}\mathbb{S}^n$ is convex and the MCF of a convex hypersurface disappears in a round point, it follows from the monotonicity properties of entropy that $\sqrt{2n}\mathbb{S}^n$ is entropy stable. In [20], Colding-Minicozzi use Theorem 3.3 to show that $\sqrt{2n}\mathbb{S}^n$ is the only compact entropy stable self-shrinker and in fact the generalized cylinders are the only entropy stable self-shrinkers. That is,

Theorem 3.5 (Colding-Minicozzi [20])**.** *If Σ is* entropy stable*, then $\Sigma = \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$.*

Colding-Minicozzi's proof requires that Σ be smooth. J. Zhu [60] extends this by showing that the presence of mild singularities does not change the conclusion.

4. Partial Classification Results for Self-shrinkers

There are many self-shrinkers and so a complete classification is likely impossible. Nevertheless, there are now many *partial* classification results – i.e., those where additional conditions, such as the index assumption of the previous section, are imposed. I will give a brief overview of what is known. We conclude with some applications of these classification results to the study of hypersurfaces of low entropy.

4.1. **Partial classification results.** In what follows, $\Sigma \subset \mathbb{R}^{n+1}$ will always be an embedded self-shrinker with $F[\Sigma] < \infty$. All classification will be done up to rotation.

For self-shrinkers of the curve shortening flow, there is a full classification.

Theorem 4.1 (Abresh-Langer [1])**.** *The only self-shrinkers $\Sigma \subset \mathbb{R}^2$ are $\mathbb{R}^1$ and $\sqrt{2}\mathbb{S}^1$.*

Recall, we restrict attention to embedded objects – there exist many other immersed self-shrinking curves. In a similar spirit, when $n = 2$, Brendle obtains very strong geometric rigidity for embedded self-shrinkers of genus-zero. This is a subtle result as there are immersed self-shrinking spheres (constructed by Drugan [23]) and also many self-shrinkers of positive genus (e.g., $\mathbb{T}^2_A$) and so both conditions must be used in the proof.

Theorem 4.2 (Brendle [12])**.** *If $\Sigma \subset \mathbb{R}^3$ has genus-zero, then $\Sigma = \mathbb{R}^2$, $\sqrt{2}\mathbb{S}^1$ or $2\mathbb{S}^2$.*

For positive genus very little is known. However, Mramor-Wang [48] have shown topological rigidity for that any closed self-shrinkers in $\mathbb{R}^3$ of positive genus.

Theorem 4.3 (Mramor-Wang [48])**.** *If $\Sigma \subset \mathbb{R}^3$ is closed and of genus $g > 0$, then Σ is isotopic to the standard embedding of a genus g surface in $\mathbb{R}^3$. In particular, if $g = 1$, then Σ is isotopic to the boundary of a small tubular neighborhood of the unit circle.*

Graphicality and mean convexity are natural conditions that are preserved by MCF and are properties that should pass, in some form, to singularity models. Self-shrinkers that have either property are geometrically very rigid.

Theorem 4.4 (Ecker-Huisken [26], L. Wang [56])**.** *If Σ can be represented as an entire graph over $\mathbb{R}^n$, then $\Sigma = \mathbb{R}^n$.*

Theorem 4.5 (Huisken [39], Colding-Minicozzi [20]). *If Σ satisfies $H_\Sigma \geq 0$, then $\Sigma = \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$.*

Both results were shown first under stronger geometric assumptions which were subsequently relaxed. For instance, Theorem 4.5 was shown by Huisken under the assumption $|A_\Sigma| \leq C$. This assumption was relaxed to $F[\Sigma] < \infty$ by Colding-Minicozzi.

Another important class of results regard the asymptotic rigidity of non-compact self-shrinkers. For instance, L. Wang showed that an asymptotically conical self-shrinker is uniquely determined by its asymptotic cone.

Theorem 4.6 (L. Wang [57]). *If Σ_1, Σ_2 are asymptotic to the same regular cone , then $\Sigma_1 = \Sigma_2$ (where defined).*

L. Wang proved this using an interesting backwards uniqueness result for the heat equation due to Escauriaza, Seregin and Šverák [27]. In particular, it also applies to incomplete Σ_i. See [4] for a purely elliptic proof. The same result holds for cylindrical ends under much stronger decay assumptions.

Theorem 4.7 (L. Wang [58]). *If Σ is asymptotic to $\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$, $0 < k < n$ at an exponential rate (Σ possibly incomplete), then $\Sigma \subset \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$.*

L. Wang gives non-complete examples showing this is sharp. L. Wang has also provided a very nice description of the ends of all embedded self-shrinkers in $\mathbb{R}^3$:

Theorem 4.8 (L. Wang [55]). *If $\Sigma \subset \mathbb{R}^3$ has finite topology (i.e., has finite genus and a finite number of ends), then each end of Σ is either asymptotic to a regular cone or to a cylinder $\sqrt{2}\mathbb{S}^1 \times \mathbb{R}$.*

In a different vein, Colding-Ilmanen-Minicozzi show interior rigidity for the cylinder:

Theorem 4.9 (Colding-Ilmanen-Minicozzi [18]). *If Σ is reasonable at infinity and is close to $\sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$, $0 < k < n$, inside of B_R for $R \gg 1$, then $\Sigma = \sqrt{2k}\mathbb{S}^k \times \mathbb{R}^{n-k}$.*

By Proposition 3.4, entropy and F-area of a self-shrinker are the same and it is natural to study low entropy self-shrinkers. The first results along this line were due to Colding-Ilmanen-Minicozzi-White [19] who show topological rigidity for low entropy closed self-shrinkers as well as prove a sharp entropy lower bound for such self-shrinkers.

Theorem 4.10 (Colding-Ilmanen-Minicozzi-White [19]). *If Σ is closed and*

$$\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}] < \lambda[\mathbb{S}^n],$$

then Σ is diffeomorphic to $\mathbb{S}^n$. Furthermore, $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$ with equality only for $\Sigma = \sqrt{2n}\mathbb{S}^n$.

This is proved using the classical MCF. Using a weak formulation of MCF, Hershkovits-White [37] have further refined the topological rigidity result for closed hypersurfaces. In another direction, L. Wang and I have shown certain topological rigidity results for low entropy asymptotically conical self-shrinkers:

Theorem 4.11 (Bernstein-L. Wang [6,7]). *If Σ satisfies $\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}]$ and is asymptotic to a regular cone, then Σ is contractible. When $n = 2$ or 3, Σ is diffeomorphic to $\mathbb{R}^n$.*

Combining Theorems 4.2, 4.10 and 4.11 gives the following complete classification of self-shrinkers in $\mathbb{R}^3$ with small entropy:

Theorem 4.12 (Bernstein-L. Wang [6]). *There is a $\delta > 0$ so that: If $\Sigma \subset \mathbb{R}^3$ satisfies $\lambda[\Sigma] \leq \lambda[\sqrt{2}\mathbb{S}^1 \times \mathbb{R}] + \delta$, then $\Sigma = \mathbb{R}^2, \sqrt{2}\mathbb{S}^1 \times \mathbb{R}$ or $2\mathbb{S}^2$. In particular, any non-flat Σ satisfies $\lambda[\Sigma] \geq \lambda[\mathbb{S}^2]$.*

We remark that Drugan-Nguyen [24] have shown that (at least one) $\mathbb{T}^n_A$ has $\lambda[\mathbb{T}^n_A] < 2$ and so the fourth lowest entropy of a shrinker in $\mathbb{R}^3$ must be achieved by a smooth surface with entropy at most that of a $\mathbb{T}^2_A$. It would interesting to know what this surface is and whether it is $\mathbb{T}^2_A$.

4.2. **Applications to low entropy hypersurfaces.** Theorems 3.5, 4.10, 4.11 and 4.12 have a number of interesting consequences for the mean curvature flow. First of all, they give sharp entropy lower bounds for all closed hypersurfaces as well as topological rigidity results for closed hypersurfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$.

Theorem 4.13 (Bernstein-L. Wang [5–7], Ketover-Zhou [46], J. Zhu [60]). *If $\Sigma \subset \mathbb{R}^{n+1}$ is closed, then*

(1) $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$;
(2) When $n = 2$ or $n = 3$ and $\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1} \times \mathbb{R}]$ then Σ is diffeomorphic to $\mathbb{S}^n$.

Another consequence is that, among closed surfaces, almost minimizers of entropy are close in the Hausdorff distance to the round sphere.

Theorem 4.14 (Bernstein-L. Wang [8], S. Wang [59]). *Given $\epsilon > 0$, there is a $\delta > 0$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is closed and has $\lambda[\Sigma] \leq \lambda[\mathbb{S}^n] + \delta$, then*

$$\inf_{\rho > 0, \mathbf{y} \in \mathbb{R}^{n+1}} \mathrm{dist}_H(\rho\Sigma + \mathbf{y}, \mathbb{S}^n) < \epsilon.$$

Somewhat surprising due to the example of $\mathbb{S}^2$ with a thin "spike". This example has F-area close to that of $\mathbb{S}^2$, but is far in Hausdorff distance.

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